Scarborough Capital Management

Asset Class Portfolio Construction Methodology

June 26, 2003

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Construction Methodology

Prepared for

Scarborough Group

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Section I – Executive Summary

Overview

This document outlines the development of asset allocation tools for Scarborough Group (Scarborough) to be used by retirement plan participants. The work product will be distributed via representatives of Scarborough Group and Scarborough Retirement Services. Ibbotson Associates (Ibbotson) developed one set of model asset class portfolios. Ibbotson analyzed broad asset classes as the basis for the work presented in this document. Ibbotson developed the following asset allocation tools:

- **Asset Class Level Portfolios** – Ibbotson provides portfolios which detail asset class allocation. The portfolios span the risk spectrum from conservative to aggressive.

About Ibbotson Associates

Ibbotson Associates opened its doors in 1977 to bridge the gap between modern financial theory and real world investment practice. Professor Roger G. Ibbotson, the company founder and Chairman, pioneered collecting the requisite data used in asset allocation and in quantifying the benefits of diversification. The firm continues to provide solutions to investment and finance problems for a diverse set of markets. Ibbotson Associates fills a growing need in the finance industry as a single-source provider of investment knowledge, expertise, and technology.

Ibbotson Associates' investment management services include asset allocation design, back-testing of portfolio strategies, mutual fund analysis, assessment of investor risk tolerance, employee training, structuring and education for retirement plans, and client-specific research. Ibbotson Associates' products include client presentation materials, computer-based training, and asset allocation software tools.

Asset Class Level Portfolios

General Comments

At the outset of this engagement, Scarborough identified the asset classes to be used for portfolio construction. In formulating diversified portfolios, Ibbotson employed a statistical technique known as optimization. The goal of optimization is to identify portfolios that maximize return for a given level of risk or minimize risk for a given level of return. Optimization requires forecasting returns, standard deviations and correlation coefficients of asset classes over the desired investing horizon. Appropriate employment of optimization as a tool also involves applying qualitative reasoning, including sensitivity analysis, assessment of probability distributions and reconciliation with investor utility and sentiment.

Findings

The table below displays the asset class level portfolios developed for Scarborough by Ibbotson.

Table 1. Asset Class Level Portfolios

Asset Class	Portfolios						
	1	2	3	4	5	6	7
Large Cap Growth Stocks	3	6	9	12	14	17	20
Large Cap Value Stocks	7	12	15	19	21	22	25
Mid Cap Stocks	2	4	6	8	10	12	13
Small Cap Growth Stocks	0	0	0	0	3	4	5
Small Cap Value Stocks	0	2	3	4	4	5	7
International Stocks	4	8	12	16	21	26	30
High Yield Bonds	6	5	4	3	0	0	0
Intermediate Term Bonds	16	15	12	10	10	8	0
Short Term Bonds	37	28	23	20	17	6	0
Cash Equivalents	25	20	16	8	0	0	0
Percent Equity	16	32	45	59	73	86	100
Percent Fixed Income	84	68	55	41	27	14	0
Expected Arithmetic Return	5.2	6.4	7.3	8.3	9.2	10.1	11.1
Expected Geometric Return	5.1	6.2	6.9	7.7	8.3	8.9	9.5
Standard Deviation	4.9	7.2	9.4	11.9	14.3	16.7	19.3

Report References

Ibbotson followed a number of steps in completing this project. Each step in the process corresponds to a section of this report. These sections are listed below:

Section II: Construction of Asset Class Level Portfolios

- **Asset Classes and Benchmarks Defined**: Identifies the specific asset classes and benchmarks considered in the project and discusses benchmark issues.

- **Mean Variance Analysis and Inputs:** Describes the methodology for forecasting the inputs used for optimization including: expected asset class returns, standard deviations and correlation coefficients.

- **Practical Considerations in Constructing Asset Class Level Portfolio Allocations:** Identifies practical methodologies applied in constructing portfolios, including portfolio constraints, portfolio spacing and performance criteria.

Section III: Appendices – Documentation and Explanation of Related Topics

Section II – Construction of Asset Class Level Portfolios

Asset Classes and Benchmarks Defined

Asset classes are defined as categories of investments with common characteristics. Some of the characteristics typically used to separate investment securities into asset classes are:

- The nature of the financial claim (debt, equity)
- The security issuer (foreign/domestic, government/corporate)
- The term length of debt obligation (short, intermediate, long)
- Others (liquidity, risk)

Ibbotson defines asset classes such that no one security can be classified into more than one category. As a result, securities within well-defined asset classes should react similarly to changes in economic circumstances.

Asset classes are comprised of a large group of securities. An index is a theoretical portfolio, created from a subset of securities in a specific asset class. Ibbotson uses indexes as benchmarks for the performance of the asset classes as a whole. In some instances, the index consists of all the securities within an asset class. Useful benchmarks are those which, represent the asset class, trade in established markets that provide price information (i.e., the New York Stock Exchange or the American Stock Exchange) and have publicly available price histories

Although there are several benchmarks available for each asset class, a set of benchmarks that best fit the criteria outlined above were chosen. The relevant asset classes and benchmarks are displayed in the following table:

Table 2. Asset Classes and Benchmarks

Asset Class	Benchmark[1,2]
Large Cap Growth Stocks	S&P/BARRA 500 Growth
Large Cap Value Stocks	S&P/BARRA 500 Value
Mid Cap Stocks	CRSP Deciles 3-5
Small Cap Growth Stocks	S&P/Barra Smallcap 600 Growth
Small Cap Value Stocks	S&P/Barra Smallcap 600 Value
International Stocks	MSCI EAFE
High Yield Bonds	LB High Yield
Intermediate Term Bonds	LB IT Gvt/Credit
Short Term Bonds	LB 1-3 Yr Gvt
Cash Equivalents	CG U.S. Domestic 3 Mo Tbill

[1] Ibbotson Associates also uses MSCI World Index, Ibbotson Associates Long-Term U.S. Government Bond Series, several CRSP Deciles and CFI (Coleman, Fisher, and Ibbotson) Synthetic Bonds to aid in determining the inputs.
[2] See Appendix A for a description of CRSP Deciles and CFI Bonds

Mean Variance Analysis and Inputs

Introduction: Required Forecasts

The methodology used by Ibbotson to determine investment portfolios is mean-variance analysis. Mean-variance analysis was developed by Harry Markowitz in the 1950's and provides a mathematical framework for generating portfolios that maximize expected return for a given level of risk (efficient portfolios), and can assist investors in making strategic asset allocation decisions.

Mean variance analysis is most appropriate for relatively efficient markets. Ibbotson assumes that asset markets are informationally efficient, with all relevant and available information fully incorporated in asset prices. If markets are informationally efficient, investor expectations (forecasts) can be discerned from market-observable data. The forecasts made by Ibbotson are not attempts to predict the market, rather attempts to ascertain the market's expectations, i.e., to determine what the market itself is forecasting.

Mean-variance analysis requires three statistical estimates for each asset class:

1. Expected return (Mean)
2. Expected risk (Standard Deviation)
3. Expected relationship between the asset classes (Correlation Coefficients)

Ibbotson develops forecasts for each of these statistics using a combination of historical data, current market information, and additional analysis. Each forecast becomes an input in mean-variance analysis.

Historical data incorporate a number of economic events and are therefore helpful to develop forecasts. Pure historical analysis, however, can be misleading. Historical data reflect the economic and market events that occurred in a specific time period. Therefore, data that do not contain the same events do not provide useful comparisons. This is problematic when the benchmarks of different asset classes have different periods of historical data available.

Ibbotson makes two important adjustments in order to properly include historical data into the forecasting process:

1. **Determine the relevant time period** – Relevant data present a good estimate of future market conditions and include outcomes that can reasonably be expected to reoccur.

2. **Adjust short-lived data** – Ibbotson estimates what might have happened had an asset class benchmark existed in prior periods. Adjustments ensure that estimates incorporate outcomes that occurred over the full, relevant time period.

In some cases evidence may exist that suggests a fundamental change in market structure, thereby limiting the efficacy of certain historical periods. Ibbotson maintains that all data from 1926 is relevant for equity asset classes. The fixed income market, however, underwent a structural change during the 1970's that makes data prior to 1970 irrelevant. During the 1970's the Bretton Woods Agreement, which fixed exchange rates, fell apart. The money supply was difficult to manage and the Federal Reserve was enticed to shift policy from managing interest rates to managing the money supply. These structural changes make the interest rate environment since 1970 inherently different from previous periods.

The following chart demonstrates how these changes have altered the risk and return characteristics of fixed income investments.

Long-Term U.S. Treasury Bonds* Historical Performance			
Period	Compound Annual Return	Average Annual Standard Deviation	Compound Annual Real Return
1926-2002	5.45	9.40	2.33
1926-1970	2.80	5.47	1.00
1970-2002	9.39	11.85	4.30

* Long-term U.S. Treasury bonds are represented by the Ibbotson Associates Long-Term U.S. Government Bond Series total return.

Ibbotson maintains that including the pre-1970 periods gives an overly conservative forecast of the risk and return characteristics of fixed income asset classes. The post-1970 period adequately captures a spectrum of returns that represent fixed income behavior through various macroeconomic and interest rate environments.

Ibbotson makes forecasts using all available relevant data. In some instances a benchmark may not have return information for the full period. In these instances Ibbotson assigns a proxy to the short-lived series. The proxy series can be used in place of or as a tool to extend the short-lived series.[3]

[3] See Appendix B for description of the construction of equity and fixed income proxies

Expected Return

Ibbotson uses the building block approach to generate expected return estimates. The building block approach uses current market statistics as its foundation and adds historical performance relationships to build expected return forecasts. This approach separates the expected return of each asset class into three components:

Building Block Component	Description
• Real Risk-Free Rate	Return that can be earned without incurring any default or inflation risk
• Expected Inflation	Additional reward demanded to compensate investors for future price increases
• Risk Premia	Additional reward demanded for accepting uncertainty associated with a given asset class

When choosing a risk-free rate, Ibbotson uses treasury yield curve rates with a maturity to match the investment period. The following table outlines the risk-free rates that are applied to various time horizons:

Table 3. Risk-Free Rates

Time Horizon	Years to Maturity	Yield*
Long-term	20	4.84%

*- All data are from the Treasury Department website as reported for March 31, 2003

The risk premia are derived from the historical relationship between the returns of the asset class and the risk-free rate, in this way, past data are incorporated into the assumption of the future returns. Various premia are added to the current risk-free rate in order to forecast the expected return unique to each asset class.

Historical returns are calculated over annual periods and may be income or total returns depending on the nature of the benchmark. In general, total returns are used for equity forecasts, whereas income returns are used for fixed income. Total return is composed of capital appreciation and income (interest payments or dividends). For fixed income asset classes, the realization of capital gains and losses is assumed to sum to zero over the time horizon of the investment.

In developing premia, the arithmetic average, as opposed to geometric average, is more appropriate for forecasting. The arithmetic average is the simple average of a return series. This measure incorporates the volatility of the returns (the risk) into expectations for the future and represents the center of the probability distribution. The geometric return is a backward looking statistic and appropriate when measuring actual historical performance.[4] Ibbotson uses the historical returns to generate forecasts; therefore arithmetic averages are used.

[4] See Appendix C for arithmetic and geometric average calculations and further description.

Domestic Equity

The building blocks Ibbotson uses to forecast domestic equity returns are the nominal risk-free rate (real risk-free rate plus expected inflation) and up to three additional risk premia: equity risk premium, size premium, and style premium. The equity risk premium measures the return equity investors require over long-term risk-free investments for the additional risk of equities. Size premium is the return that investors demand for holding stock in companies with market capitalizations different than the benchmark used to calculate the equity risk premium. Style premium is the additional return demanded for holding equities with different style (growth or value) than the broad asset class benchmark.

The following box displays the general formulae for calculating the equity building blocks:

Equity Risk Premium[5]	=	CRSP Deciles 1-2 Total Return	– Long-Term Government Bond Income Return	– Growth in P/E Ratio
Size Premium	=	Smaller Capitalization Domestic Equity Benchmark Total Return	– CRSP Deciles 1-2 Total Return	
Style Premium	=	Growth or Value Index Total Return	– (50% Growth Index Total Return 50% Value Index Total Return)	

[5] See Appendix E for an explanation on the Supply Side Equity Risk Premium

International Equity

The foundation for the expected returns of the international equity asset classes is the international equity risk premium. The international equity risk premium measures the reward investors receive for holding assets that are domiciled outside the United States. Due to the lack of performance history for international benchmarks, the premium cannot be calculated by using the difference between historical returns with sufficient confidence. As a result, Ibbotson uses the International Capital Asset Pricing Model (CAPM) to calculate the international equity risk premium.

To use CAPM it is first necessary to calculate the world equity risk premium. This premium is the incremental reward demanded by investors for holding a complete basket of risky assets worldwide over the U.S. risk-free rate. This relationship is computed indirectly. The world equity risk premium is derived by dividing the domestic equity risk premium by the sensitivity of domestic markets to the world equity market. This method provides a stable "anchor" for the world equity risk premium because it utilizes the domestic equity risk premium, which implies a data history extending back to 1926.

Using the world equity risk premium and the relationship between international markets and worldwide markets, Ibbotson calculates the international equity risk premium. This removes the market effects of the United States from the premium. The international equity risk premium is calculated by multiplying the world equity risk premium by the sensitivity of international markets to the world equity market.

The relationships previously mentioned are calculated using regression analysis. Ibbotson identifies the relationship between domestic and world markets by regressing the monthly returns of domestic large-cap equity against the monthly returns of equity worldwide (including the United States). The benchmark for domestic large-cap equity is the CRSP Deciles 1-2, and the benchmark for world equity is the MSCI World Index. The domestic equity risk premium is divided by this resulting beta in order to approximate the world equity risk premium.

The relationship between international markets and worldwide markets is found by regressing the monthly returns of international equity against the monthly returns of world equity. The benchmark for international equity is the MSCI EAFE Index. This beta is then multiplied by the world equity risk premium in order to approximate the international equity risk premium.

$$\text{Domestic Equity Risk Premium} = \text{World Equity Risk Premium} \times \text{Beta}_{\text{US vs. World}}$$

$$\text{World Equity Risk Premium} = \frac{\text{Domestic Equity Risk Premium}}{\text{Beta}_{\text{US vs. World}}}$$

$$\text{International Equity Risk Premium} = \text{World Equity Risk Premium} \times \text{Beta}_{\text{Int'l vs. World}}$$

$$\text{International Equity Risk Premium} = \frac{\text{Domestic Equity Risk Premium}}{\text{Beta}_{\text{US vs. World}}} \times \text{Beta}_{\text{Int'l vs. World}}$$

Ibbotson calculates beta using the following linear regressions:

$$\text{Large-Cap Equity} = A + \text{Beta}_{\text{US vs. World}} \times (\text{World Stocks})$$

$$\text{International Equity} = A + \text{Beta}_{\text{Intl vs. World}} \times (\text{World Stocks})$$

Fixed Income

Ibbotson may apply a horizon premium and default premium to the nominal risk-free rate when developing forecasts for fixed income expected returns. The horizon premium measures the excess yield long-term fixed income investors expect to receive in exchange for accepting additional uncertainty and potential loss of liquidity. Ibbotson estimates horizon premium as the difference (in the income return) between two government bonds. The first government bond has the same maturity as the asset class being modeled (the government bond proxy), the second government bond is the risk-free rate. The corporate default premium measures the historical reward demanded for holding corporate bonds rather than government bonds of the same maturity. The corporate default premium is equal to the difference between a pure corporate benchmark and a government bond of the same maturity. This difference is multiplied by the corporate exposure in the asset class.

Each of the specific fixed income premia calculations are outlined in the box below:

Horizon Premium = CFI Government Bond Proxy[A] Income Return − CFI Government Bond Proxy[B] Income Return

Corporate Default Premium = (Corporate Bond Index Income Return − CFI Government Bond Proxy[A] Income Return) X % Corporate Bond Exposure

[A] Same maturity (average or current) as the asset class benchmark
[B] Same maturity as the time horizon, i.e. 20 years

Standard Deviation

Mean-variance analysis requires a quantifiable measure of risk for each asset class. Ibbotson forecasts standard deviation as an estimate that risk. Standard deviation measures dispersion around an average return.

Ibbotson uses historical data to forecast standard deviation because it provides an unbiased estimate of future volatility. Ideally, Ibbotson uses historical standard deviations using all available and relevant data (1926 and 1970 for equity and fixed income, respectively).

Ibbotson uses the *ratio method* to extend the standard deviation estimates of the shorter-lived asset class benchmarks so that they incorporate all relevant economic events. The ratio method attempts to extend the standard deviation estimate for certain asset class benchmarks using two data series:

1. **Short benchmark** – an asset class benchmark used in this project that does not have historical data over the full, relevant time period.

2. **Long proxy** – an index that has historical data over the full, relevant time period and is economically similar to the short benchmark, i.e. there is a logical reason to believe that the returns of the two series are highly related.

Ibbotson finds the ratio between the standard deviation of the short benchmark and that of the long proxy over a common time. The common time period is the inception of the short benchmark (unless otherwise noted). Ibbotson assumes that the relationship between the short benchmark and long proxy is representative of what it would have been if both series had existed over the full relevant time period.

The ratio is multiplied by the standard deviation of the long proxy measured over the full, relevant period. The product is an estimate of what the standard deviation of the short benchmark would have been had it existed over the full, relevant period.

$$\text{Extended Standard Deviation} = \frac{\text{Std Dev of Short Benchmark}_{\text{Short}}}{\text{Std Dev of Long Proxy}_{\text{Short}}} \times \text{Std Dev of Long Proxy}_{\text{Long}}$$

Correlation Coefficient

In the mean variance analysis setting, the standard deviation of a portfolio is based not only on the risk of each asset class, but on the relationship between the returns of asset classes as well. The relationship between the returns of asset classes is measured by the correlation coefficient.

The correlation coefficient measures the degree to which two asset classes' returns change with respect to each other. The statistic can range between positive one (+1) and negative one (-1) and provides the following information about the relationship between asset classes:

- **Positive one (+1):** perfect positive relationship – two assets classes move together in the same direction.
- **Negative one (-1):** perfect negative relationship – two asset classes move together in opposite directions.
- **Zero (0):** no relationship – the movements of two asset classes are unrelated.

Ibbotson typically uses correlation coefficients derived from the historical returns of the asset class benchmarks. Ibbotson prefers to use equity to equity correlation coefficients from 1926 forward, and all correlations to fixed income asset classes from 1970 forward. Correlation coefficients must be extended for series that do not have history for the full relevant period.

A sophisticated statistical process extends asset class benchmarks that do not have complete data histories, (i.e. since 1926 for equities and 1970 for fixed income) but do have a relatively high correlation coefficient with another proxy. This estimate is an approximation of what the correlation coefficient between the two series might have been if both had existed over the longer time period.[6]

Most asset classes with benchmarks that do not cover the full relevant time period (1926-2002 for equity and 1970-2002 for bonds) have their correlation coefficients with other, select asset classes determined by this process. However, some asset classes are do not have their correlations extended, because Ibbotson has determined that there are no appropriate benchmarks with which to extend. The following is a complete list of the asset classes for which Ibbotson does not extend their correlations and the corresponding date from which the correlations are calculated:

Asset Class	Data Start Date
International Stocks	1970

The following pages present the input development tables and final results of the input construction.

[6] See *Appendix D-Correlation Extension* for further details.

Input Construction Results

This section presents the forecasted inputs Ibbotson used to develop the portfolios in this project. Ibbotson used the inputs below when developing the asset class level portfolios:

Table 4. Inputs Summary

Asset Class	Benchmark	Expected Returns	Standard Deviation
Large Cap Growth Stocks	S&P/BARRA 500 Growth	9.94	24.10
Large Cap Value Stocks	S&P/BARRA 500 Value	11.22	19.12
Mid Cap Stocks	CRSP Deciles 3-5	12.20	25.08
Small Cap Growth Stocks	S&P/Barra Smallcap 600 Growth	10.61	21.17
Small Cap Value Stocks	S&P/Barra Smallcap 600 Value	15.61	26.93
International Stocks	MSCI EAFE	10.39	24.80
High Yield Bonds	LB High Yield	6.55	12.58
Intermediate Term Bonds	LB IT Gvt/Credit	4.56	5.56
Short Term Bonds	LB 1-3 Yr Gvt	4.07	4.01
Cash Equivalents	CG U.S. Domestic 3 Mo Tbill	3.47	2.77

	Expected Correlation Coefficients									
	Large Cap Growth	Large Cap Value	Mid Cap	Small Cap Growth	Small Cap Value	Intl Stocks	High Yield	IT Bonds	ST Bonds	Cash Equiv.
Large Cap Growth Stocks	1.00	0.81	0.89	0.79	0.57	0.48	0.47	0.20	0.20	0.08
Large Cap Value Stocks	0.81	1.00	0.86	0.86	0.82	0.46	0.45	0.20	0.18	0.07
Mid Cap Stocks	0.89	0.86	1.00	0.96	0.79	0.43	0.50	0.15	0.16	0.08
Small Cap Growth Stocks	0.79	0.86	0.96	1.00	0.84	0.37	0.47	0.11	0.13	0.07
Small Cap Value Stocks	0.57	0.82	0.79	0.84	1.00	0.30	0.39	0.10	0.09	0.05
International Stocks	0.48	0.46	0.43	0.37	0.30	1.00	0.35	-0.03	-0.07	-0.04
High Yield Bonds	0.47	0.45	0.50	0.47	0.39	0.35	1.00	0.54	0.48	0.04
Intermediate Term Bonds	0.20	0.20	0.15	0.11	0.10	-0.03	0.54	1.00	0.93	0.36
Short Term Bonds	0.20	0.18	0.16	0.13	0.09	-0.07	0.48	0.93	1.00	0.64
Cash Equivalents	0.08	0.07	0.08	0.07	0.05	-0.04	0.04	0.36	0.64	1.00

Developing Expected Returns

The expected return of each asset class is the sum of the risk-free rate and the associated premia.

Table 5. Expected Return Development

Asset Class	Benchmark	Risk-Free Rate	Equity Premium				Fixed Income Premium		Expected Return
			Equity Risk Premium	Size Premium	Style Premium	Intl Risk Premium	Horizon Premium	Default Premium	
Large Cap Growth Stocks	S&P/BARRA 500 Growth	4.84	5.01	0.73	(0.64)	-	-	-	**9.94**
Large Cap Value Stocks	S&P/BARRA 500 Value	4.84	5.01	0.73	0.64	-	-	-	**11.22**
Mid Cap Stocks	CRSP Deciles 3-5	4.84	5.01	2.35	-	-	-	-	**12.20**
Small Cap Growth Stocks	S&P/Barra Smallcap 600 Growth	4.84	5.01	3.26	(2.50)	-	-	-	**10.61**
Small Cap Value Stocks	S&P/Barra Smallcap 600 Value	4.84	5.01	3.26	2.50	-	-	-	**15.61**
International Stocks	MSCI EAFE	4.84	-	-	-	5.55	-	-	**10.39**
High Yield Bonds	LB High Yield	4.84	-	-	-	-	(0.11)	1.82	**6.55**
Intermediate Term Bonds	LB IT Gvt/Credit	4.84	-	-	-	-	(0.51)	0.23	**4.56**
Short Term Bonds	LB 1-3 Yr Gvt	4.84	-	-	-	-	(0.77)	-	**4.07**
Cash Equivalents	CG U.S. Domestic 3 Mo Tbill	4.84	-	-	-	-	(1.37)	-	**3.47**

Developing Standard Deviations

Ibbotson uses historical data and the ratio method to calculate standard deviation estimates. If the asset class benchmark does not extend over the full, relevant period, the expected standard deviation (σ) is equal to column A divided by column B multiplied by column C.

Table 6. Standard Deviation Development

Asset Class	Benchmark	Benchmark		Proxy Information				Expected σ (A/B)*C
		Inception Date	Annual σ (A)	Long Proxy	Inception Date	σ for Common Period (B)	σ for Relevant Period (C)	
Large Cap Growth Stocks	S&P/BARRA 500 Growth	1975	19.38	S&P 500	1926	16.48	20.49	**24.10**
Large Cap Value Stocks	S&P/BARRA 500 Value	1975	15.38	S&P 500	1926	16.48	20.49	**19.12**
Mid Cap Stocks	CRSP Deciles 3-5	1926	25.08	N/A	N/A	N/A	N/A	**25.08**
Small Cap Growth Stocks	S&P/Barra Smallcap 600 Growth	1994	14.08	EEP: S&P SmallCap 600	1926	18.66	28.06	**21.17**
Small Cap Value Stocks	S&P/Barra Smallcap 600 Value	1994	17.91	EEP: S&P SmallCap 600	1926	18.66	28.06	**26.93**

International Stocks	MSCI EAFE	1970	22.62	CRSP Deciles 1-2	1926	17.97	19.70	24.80
High Yield Bonds	LB High Yield	1970	12.58	N/A	N/A	N/A	N/A	12.58
Intermediate Term Bonds	LB IT Gvt/Credit	1973	5.62	GBP: LB IT Gvt/Credit	1970	5.48	5.42	5.56
Short Term Bonds	LB 1-3 Yr Gvt	1976	4.24	GBP: LB 1-3 Yr Gvt	1970	4.18	3.95	4.01
Cash Equivalents	CG U.S. Domestic 3 Mo Tbill	1970	2.77	N/A	N/A	N/A	N/A	2.77

Practical Considerations in Constructing Asset Class Level Portfolio Allocations

When creating portfolios at the asset class level, Ibbotson focuses on two major qualifications: (1) efficiency from a mean-variance perspective, and (2) investor preferences. Portfolios that provide the best risk/return characteristics may not be acceptable to many clients due to counterintuitive allocations and investment biases. Furthermore, the most quantitatively efficient portfolios may not take into account possible errors in the input forecast. All of these factors are incorporated into the portfolio recommendations. The following sections: Ibbotson Constraints, Forecast Performance Criteria, and Portfolio Spacing provide an overview of how Ibbotson approaches portfolio construction within the mean-variance setting.

Ibbotson Constraints

Performing an unconstrained mean-variance optimization will often result in asset allocations that are not deemed practical by the investor and the investment professional. In an ideal world, the inputs used in mean-variance analysis would perfectly reflect future asset class behavior and would result in efficient portfolios that also meet investor "tolerances" for asset holdings. Unfortunately, this is not the case. Regardless of one's method for calculating mean-variance analysis inputs, there will be instances where the resulting values differ dramatically from more qualitative expectations and investor tolerances. In addition, short-lived data series may result in unstable inputs. In both of these cases, it is necessary to constrain the allocations to such asset classes to reflect qualitative information, investor tolerances, or the lower confidence in various asset class inputs.

Growth and Value Equity

Empirical evidence shows that value stocks have consistently outperformed growth stocks over long periods of time. Value stocks have not only provided higher returns than growth stocks, but have also exhibited less volatility. The following table shows value's historical out-performance over the longest data period available:

Annual Data 1975 – 2002	Arithmetic Mean	Standard Deviation	Sharpe Ratio
S&P/Barra 500 Growth	13.86	19.38	0.72
S&P/Barra 500 Value	15.14	15.38	0.98
Annual Data 1978 – 2002			
Wilshire Target MidCap Growth	14.40	18.75	0.77
Wilshire Target MidCap Value	16.83	16.98	0.99
Annual Data 1978 – 2002			
Wilshire Target Small Growth	14.47	20.16	0.72
Wilshire Target Small Value	17.51	17.80	0.98

Therefore, Ibbotson allocates a relatively higher percentage to value stocks than to growth stocks. Based upon expectations of risk and return, value should receive the entire allocation. There seems to be persistence and predictability to the value dominance phenomenon. However, Ibbotson cannot say with certainty that this market inefficiency will continue over the long-term. Additionally, many investors would feel uncomfortable with eliminating growth stocks and managers from their portfolio.

There is no ideal rule of thumb as to the appropriate allocation to growth stocks. A market neutral portfolio would contain 50% value stocks and 50% growth stocks. We recommend a value tilt of approximately 65% value and 35% growth beginning with the conservative portfolio and increasing towards an equal market weighted percentage of growth and value as the portfolios become more aggressive where risk is more acceptable.

International Stocks

Ibbotson considers a number of quantitative and qualitative factors when allocating to international stocks. Although other aspects are considered, the base of Ibbotson's allocation policy is a historical analysis of portfolios that include various levels of international stocks.

Ibbotson uses a simplified setting to examine the role of international stocks in a portfolio. This example makes the following assumptions:

- The overall allocation to equity is 60 percent and the allocation to bonds is 40 percent.
- Only the following three asset classes are considered:[7]
    1. Large-cap stocks
    2. Intermediate-term bonds
    3. International stocks
- The inputs used in mean-variance analysis are Ibbotson's expected returns, standard deviations, and correlations as constructed using the Ibbotson building block methodology.

In this example, Ibbotson divides the total equity allocation between domestic equity (large-cap stocks) and international stocks. The example below shows how various allocations to international stocks effects the overall risk and return characteristics of the portfolio:

Allocation to International Stocks	Percent of Equity	Expected Return	Standard Deviation	Sharpe Ratio[8]
0%	0%	8.10%	13.14%	0.62
12	20	8.08	12.57	0.64
15	25	8.07	12.51	0.65
18	30	8.06	12.47	0.65
21	35	8.06	12.47	0.65
24	40	8.05	12.50	0.64
27	45	8.05	12.56	0.64
30	50	8.04	12.66	0.64

This analysis shows that the addition of international equities to the portfolio actually reduces the volatility despite the higher level of international volatility as a stand-alone asset class. This occurs because of its lower correlation with domestic equities and bonds. The portfolio volatility decreases until international equity is approximately 35 percent of total equity. Above 35 percent, risk increases as the volatility of the international equity asset class begins to overcome the correlation benefit. The portfolio

[7] The S&P 500, Ibbotson Intermediate-Term Government Bond, and the MSCI EAFE are the benchmarks for large-cap stocks, intermediate-term bonds, and international stocks, respectively.

[8] The Sharpe ratio is excess return divided by standard deviation. Excess return is measured as the portfolio return minus the risk-free rate. In this example, the risk-free rate is assumed to be zero.

return per incremental unit of risk (Sharpe Ratio) begins to decrease when international equity is 40 percent or more of total equity.

Based on these results, one could argue that the appropriate allocation to international stocks is between 25 to 35 percent of total equity. However, Ibbotson prefers an international allocation that is approximately 25 to 30 percent of total equity.

Ibbotson limits the international stock allocation because the historical data overstates the return of this asset class. Over the period analyzed, the dollar depreciated against many major foreign currencies, especially the Japanese yen. This currency movement resulted in returns above what would be predicted by the asset class's systematic risk (as measured by the International CAPM). Ibbotson, however, does not necessarily predict a similar degree of currency movement in the future. The forecasted return incorporates a slightly higher premium over domestic equities.

International equity is primarily a diversification tool. It offers different opportunities than domestic equity in terms of industrial composition and expectations for economic growth (across both regions and countries). The primary diversification benefits of international stocks are due to low correlation coefficients with domestic asset classes. This benefit applies to portfolios across the spectrum of risk levels (conservative through aggressive). Consequently, historical analysis for different combinations of equities and bonds provides similar results.

Ibbotson bases the appropriate asset allocation to international stocks on historical data and expectations. The exact allocation, however, is somewhat subjective. Therefore, as a general rule Ibbotson has an international stocks target allocation of about 25 to 30 percent of total equities across all portfolios, the percentage will depend of where the portfolio lies along the risk spectrum.

Small-Cap Stocks

Ibbotson uses historical analysis to examine the optimal allocation to small-cap stocks. The historical risk and return characteristics of small-cap stocks, large-cap stocks, and intermediate-term bonds are examined over the 1970-2002 period.[9]

The results of this analysis do not point to a single optimal allocation. The addition of small-cap stocks does not reduce total portfolio risk at any level. However, increasing the allocation to small-cap stocks does not decisively reduce the portfolio's risk-adjusted return (Sharpe ratio) by increasing total portfolio risk.

The following table shows the results of this analysis for 60 percent equity, 40 percent bonds portfolio.

Allocation to Small-Cap Stocks	Percent of Equity	Arithmetic Mean Return	Standard Deviation	Sharpe Ratio
0%	0%	8.10	13.14	0.62
12	20	8.46	13.64	0.62
15	25	8.55	13.82	0.62
18	30	8.64	14.02	0.62
21	35	8.73	14.23	0.61
24	40	8.83	14.47	0.61
27	45	8.92	14.72	0.61
30	50	9.01	14.99	0.60

This example shows that portfolio return increases with portfolio risk. The appropriate small-cap allocation depends on the desired risk characteristics of the portfolio. Ibbotson assumes that a portfolio's return and risk should increase with an investor's risk tolerance. Ibbotson increases small-cap allocations (as a percentage of total stocks) when developing more aggressive portfolios. The allocation to small-cap stocks is dependent on the intended risk of the total portfolio.

In the conservative portfolios, Ibbotson does not allocate to small-cap stocks. In the moderate portfolios, this allocation is approximately 25 percent of total domestic equity. In the most aggressive portfolios, Ibbotson may allocate up to 30 percent of domestic equity to small-cap stocks.

Ibbotson uses historical performance and a market-neutral strategy as the basis for the small-cap allocations. Small-cap stocks represent approximately 15 to 20 percent of the total domestic market capitalization. A market-neutral strategy would have small-cap allocations that match this asset class's market share. Historically, small-cap equities have consistently demonstrated returns greater than what would be predicted by the asset class's systematic risk (as measured by the CAPM). This makes small-cap stocks an attractive asset class. Ibbotson tends to overweight this asset class (relative to a market-neutral strategy) in the moderate through aggressive portfolios.

[9] The CRSP Deciles 6-8, S&P 500, and Ibbotson Intermediate-Term Government Bond are the benchmarks for small-cap equity, large-cap equity, and intermediate-term fixed income, respectively.

High-Yield Bonds

The allocation to high yield bonds is based on the risk and return characteristics of this asset class. High-yield bonds are investments in low-grade bonds and the payment of interest and principal is partially contingent on the fortunes of the issuing company. This asset class tends to have both equity and bond qualities. High yield does not receive a major allocation because it is possible to achieve similar risk and return characteristics using a mixture of stocks and bonds. Placing high-yield bonds into a portfolio may reduce the equity allocation and the overall efficiency of the portfolio.

There are periods, however, where high-yield bonds can offer some diversification benefits. The potential diversification benefits justify the small high-yield allocation seen in the portfolios. Ibbotson generally will not exceed high-yield allocations of five to ten percent of the total portfolio.

Forecast Performance Criteria

The following is a brief description of the performance criteria for conservative, moderate, and aggressive portfolios. These descriptions focus on short-term volatility and chance of loss, investor risk aversion and the related ability to stay the course, as well as wealth protection and real growth measured by the potential of an investment to outpace inflation in both the short and long run. These qualities are not intended to be a complete list of absolute requirements, but a list of the items Ibbotson takes into consideration when developing model portfolios.

The Conservative Investor

The conservative investor is particularly sensitive to short-term losses, but still has the goal of beating expected inflation over the long run. A conservative investor's aversion to short-term losses could compel them to shift into the most conservative investment if the losses occur. Conservative investors would accept lower long-term return in exchange for smaller and less frequent changes in portfolio value. Analyzing the risk-return choices available, a conservative investor is usually willing to accept a lower return in order to assure safety of his or her investment. The following criteria help to ensure that such investors have the best chance of achieving these goals:

- The portfolio should have *approximately* a 90 percent chance of achieving a non-negative return over a one-year holding period. This accounts for substantial risk aversion; any stricter criteria used to avoid risk (such as a 95 percent chance) would significantly limit the intermediate and long-term upside potential of the portfolio.

- The portfolio should have at least a 75 percent chance of keeping pace with expected inflation over a three-year span and 90 percent chance over a five-year span. Using a model developed by Ibbotson, an expected inflation rate of approximately 1.5 percent has been forecast for the next three to five years.[10]

- Investment periodicals (both retail and institutional) often single out three and five years as relevant time periods for manager selection and evaluation. Investors will tend to examine a portfolio's performance to date and re-evaluate their investment decision over these periods. Ibbotson assumes that a high probability of achieving the minimal investment goal of keeping pace with inflation is desired over these periods.

- The portfolio should have an expected return (refers to the expected value) that outpaces expected inflation by at least three percent over a 20-year holding period. An expected inflation rate range of 2-3 percent has been forecast over the next 20 years. There should be growth in the real value of assets over the long run.

The Moderate Investor

The moderate investor is willing to accept more risk than the conservative investor, but is probably not willing to accept the short-term risk associated with achieving a long-term return substantially above the inflation rate. A moderate investor is somewhat concerned with short-term losses and would shift to a more conservative option in the event of significant short-term losses. The safeties of investment and return are of equal importance to the moderate investor. Given these preferences, Ibbotson constructs the moderate portfolio to meet the following criteria:

- The portfolio should have a 75 percent chance of achieving a non-negative return over a one-year time frame and at *approximately* a 90 percent chance over a three-year holding period. These constraints are more liberal than those for the conservative portfolio, but still account for moderate short-term loss aversion.

- The portfolio should have at least a 75 percent chance of keeping pace with expected inflation over a three-year holding period and 90 percent chance over a five-year span. An expected inflation rate of approximately 1.5 percent has been forecast for the next three to five years. This is the same as the criteria for the conservative and aggressive investors.

[10] See *Appendix F* for details on this model.

- The portfolio should be expected to outpace expected inflation by *approximately* 5 percent over a 20-year holding period. An expected inflation rate range of 2-3 percent has been forecast over the next 20 years.

The Aggressive Investor

The aggressive portfolio should be constructed with the goal of maximizing long-term expected returns rather than to minimize possible short-term losses. The aggressive investor values high returns relatively more and can tolerate both large and frequent fluctuations in portfolio value in exchange for a higher return. In order to construct such a portfolio, Ibbotson designs a long-term aggressive portfolio to meet the following criteria:

- The portfolio should have a 75 percent chance of achieving a non-negative return over a three-year holding period. There should still be some marginal protection against risk, but a lesser degree of concern with short-term loss potential.

- The portfolio should have at least a 75 percent chance of keeping pace with expected inflation over the three-year holding period and 90 percent chance over a five year span. An expected inflation rate of approximately 1.5 percent has been forecast for the next three to five years. This is the same as the criteria for the conservative and moderate investors.

- The portfolio should be expected to beat expected inflation by approximately 7 percent over a 20-year holding period. An expected inflation rate range of 2-3 percent has been forecast over the next 20 years. The expected long-term real return is approximately three times that of the conservative portfolio, and one and a half that of the moderate portfolio.

Portfolio Spacing

The portfolios developed for Scarborough are the result of the Ibbotson inputs construction methodology, the relative constraints used to reflect prudent and practical investor considerations, and the meeting of basic performance criteria. Other considerations, such as portfolio "spacing" are also reflected in these allocations. Portfolio spacing refers to the change in standard deviation (risk) from one portfolio to the next. The goal is to ensure that the risk spread between each portfolio is relatively equal (i.e., there is no benefit in offering five portfolios if they all have similar risk characteristics). Because standard deviation estimations are based on historical data, they are more stable than nominal return estimations (which rely partly on current inflation expectations as expressed in the treasury yield curve). As a result, Ibbotson prefers to base "spacing" upon the variable that will change least from year to year. This helps to ensure that the target portfolios will not experience a drastic shift in asset class weightings from one period to the next.

Probability Distributions

Ibbotson uses a number of quantitative analyses to assist in the portfolio development process. In order to quantify future return prospects for a portfolio, a lognormal distribution is constructed (using an asset mixes' expected return and volatility). The result is a probability distribution of future returns. The percentiles that are generated represent the probabilities associated with achieving various compound annual returns over multiple time horizons, and therefore, accounts for the effects of time horizon on risk.

The following pages contain tables that represent the probability distributions of portfolio mixes. This distribution is skewed to the right with the expected value being greater than the median. Furthermore, returns cannot fall below negative 100 percent. These properties make lognormal distribution more representative of the behavior of market returns.

The probability distribution tables display future return prospects. All probability distributions are expressed in terms of compound annual returns. The expected value represents the probability weighted center of the return distribution, while the 50th percentile is the median return with half of the possible compound annual returns above it and half of the possible compound annual returns below it.

Probability Distributions:

Probability Distributions

Portfolio 1

	1 Year	3 Years	5 Years	10 Years	20 Years
95th Percentile	13.4	9.8	8.7	7.7	6.9
90th Percentile	11.5	8.8	7.9	7.1	6.5
75th Percentile	8.4	7.0	6.6	6.1	5.8
Expected Value	5.2	5.1	5.1	5.1	5.1
50th Percentile	5.1	5.1	5.1	5.1	5.1
25th Percentile	1.8	3.2	3.6	4.0	4.4
10th Percentile	-1.0	1.5	2.3	3.1	3.7
5th Percentile	-2.7	0.5	1.5	2.6	3.3

Portfolio 2

	1 Year	3 Years	5 Years	10 Years	20 Years
95th Percentile	18.6	13.2	11.6	10.0	8.8
90th Percentile	15.8	11.6	10.4	9.1	8.2
75th Percentile	11.1	9.0	8.3	7.7	7.2
Expected Value	6.4	6.2	6.2	6.2	6.2
50th Percentile	6.2	6.2	6.2	6.2	6.2
25th Percentile	1.4	3.4	4.0	4.6	5.1
10th Percentile	-2.7	1.0	2.1	3.3	4.1
5th Percentile	-5.0	-0.4	1.0	2.5	3.6

Portfolio 3

	1 Year	3 Years	5 Years	10 Years	20 Years
95th Percentile	23.4	16.1	14.0	11.9	10.4
90th Percentile	19.6	14.0	12.4	10.7	9.6
75th Percentile	13.4	10.6	9.7	8.9	8.3
Expected Value	7.3	7.0	7.0	6.9	6.9
50th Percentile	6.9	6.9	6.9	6.9	6.9
25th Percentile	0.8	3.3	4.1	4.9	5.5
10th Percentile	-4.4	0.2	1.7	3.2	4.2
5th Percentile	-7.4	-1.6	0.2	2.1	3.5

Portfolio 4

	1 Year	3 Years	5 Years	10 Years	20 Years
95th Percentile	28.9	19.5	16.7	14.0	12.1
90th Percentile	23.9	16.7	14.6	12.5	11.1
75th Percentile	15.9	12.3	11.3	10.2	9.4
Expected Value	8.3	7.9	7.8	7.7	7.7
50th Percentile	7.7	7.7	7.7	7.7	7.7
25th Percentile	0.0	3.2	4.2	5.2	5.9
10th Percentile	-6.4	-0.7	1.1	3.0	4.3
5th Percentile	-10.1	-3.0	-0.7	1.7	3.4

Portfolio 5

	1 Year	3 Years	5 Years	10 Years	20 Years
95th Percentile	34.2	22.5	19.2	15.9	13.6
90th Percentile	28.0	19.2	16.7	14.2	12.4
75th Percentile	18.2	13.9	12.6	11.3	10.4
Expected Value	9.2	8.6	8.5	8.4	8.3
50th Percentile	8.3	8.3	8.3	8.3	8.3
25th Percentile	-0.8	2.9	4.1	5.3	6.2
10th Percentile	-8.4	-1.7	0.5	2.7	4.3
5th Percentile	-12.6	-4.3	-1.6	1.2	3.2

Portfolio 6

	1 Year	3 Years	5 Years	10 Years	20 Years
95th Percentile	39.5	25.6	21.6	17.7	15.1
90th Percentile	32.1	21.7	18.7	15.7	13.7
75th Percentile	20.5	15.4	13.9	12.4	11.4
Expected Value	10.1	9.3	9.1	9.0	8.9
50th Percentile	8.9	8.9	8.9	8.9	8.9
25th Percentile	-1.7	2.6	4.0	5.4	6.4
10th Percentile	-10.3	-2.6	-0.2	2.4	4.3
5th Percentile	-15.1	-5.7	-2.6	0.6	3.0

Portfolio 7

	1 Year	3 Years	5 Years	10 Years	20 Years
95th Percentile	45.4	28.9	24.3	19.7	16.6
90th Percentile	36.5	24.4	20.8	17.4	15.0
75th Percentile	23.0	17.1	15.3	13.6	12.3
Expected Value	11.1	10.0	9.8	9.6	9.5
50th Percentile	9.5	9.5	9.5	9.5	9.5
25th Percentile	-2.6	2.4	3.9	5.5	6.7
10th Percentile	-12.2	-3.7	-0.8	2.1	4.2
5th Percentile	-17.6	-7.1	-3.6	0.1	2.7

Sensitivity Analysis

Ibbotson's asset allocation inputs are based in part upon historical measurements intended to capture the average performance for asset class premia and volatility over various economic and market cycles. Since the future may not conform to our model, sensitivity analysis is employed to evaluate the stability of the portfolio mixes' performance through a variety of alternative input assumptions. What is important in sensitivity analysis is that reasonable changes in the inputs do not significantly alter the portfolios' proximity to the efficient frontier. In other words, the portfolios do not deviate dramatically from established risk targets (although a certain degree of risk deviation is unavoidable).

The scenarios are purely hypothetical situations that can be used to test the stability of the portfolios under different environments. Although the risk/return characteristics of the asset classes change under each different scenario, the portfolios follow a consistent asset allocation strategy. Changes in the input assumptions, however, alter the face of the efficient frontier. The absolute changes in the inputs are not as important as the proximity of the portfolios relative to the new efficient frontier under each different scenario.

The following pages provide a summary of the variations in the inputs followed by graphs of the new efficient frontiers. The effects of the changes in the inputs are detailed in the graph. The graph plots the model portfolios and the corresponding portfolios with the same standard deviation as the base case. The table below outlines the points shown on each frontier:

Portfolios	Same Allocation as Base Case	Same Risk Level as Base Case
Ultra Conservative	1	A
Conservative	2	B
Moderate-Conservative	3	C
Moderate	4	D
Moderate-Aggressive	5	E
Aggressive	6	F
Ultra Aggressive	7	G

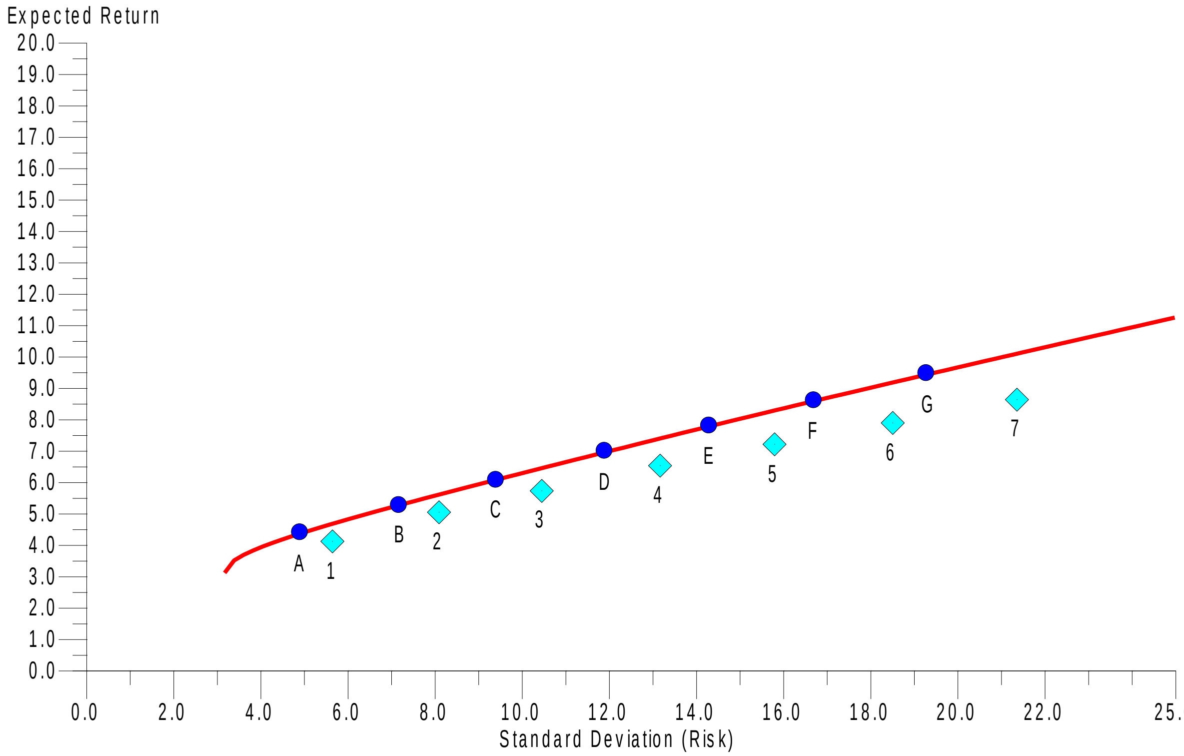
Points 1 through 7 use the display the asset allocations of the portfolios developed in this project. Points A through G are the efficient points on each new frontier that have the same standard deviation as the portfolios under the base case assumptions. The horizontal distance between corresponding points (i.e. portfolio 1 and portfolio A, etc.) illustrates the change in risk level due to changes in the input assumptions.

A summary table of variations in the inputs and a frontier graph detailing the effect changes in the inputs have on the allocations for each scenario are provided on the following pages.

Scenario 1 – Expected returns decrease and standard deviations increase for all asset classes

Asset Class	Change in E(R)	Change in Standard Deviation
Large-Cap Growth Stocks	-2.0	2.0
Large-Cap Value Stocks	-2.0	2.0
Mid-Cap Stocks	-2.5	2.5
Small-Cap Growth	-3.0	3.0
Small-Cap Value	-3.0	3.0
International Stocks	-3.0	3.0
High Yield	-2.0	2.0
Intermediate-term Bonds	-1.0	1.0
Short-term Bonds	-1.0	1.0
Cash Equivalents	-0.5	0.5

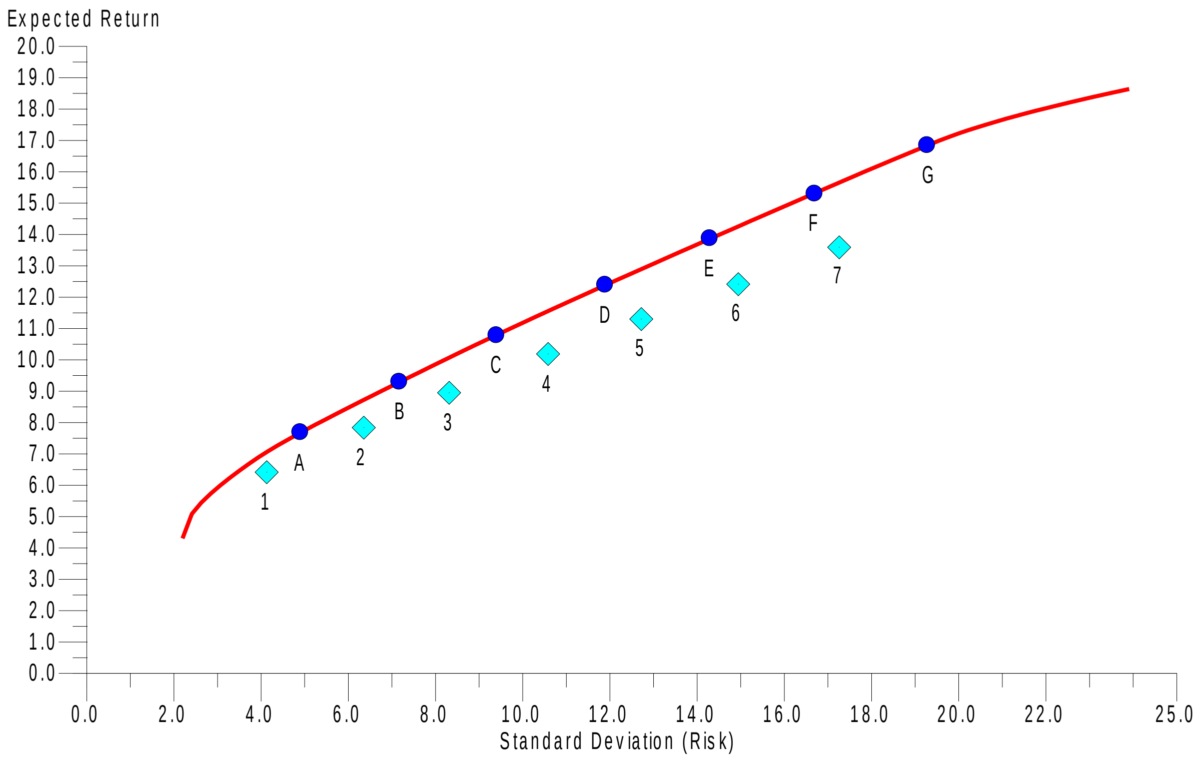
Correlation coefficients remain unchanged



Scenario 2: Expected returns increase and standard deviations decrease for all asset classes

Asset Class	Change in E(R)	Change in Standard Deviation
Large-Cap Growth Stocks	2.0	-2.0
Large-Cap Value Stocks	2.0	-2.0
Mid-Cap Stocks	2.5	-2.5
Small-Cap Growth	3.0	-3.0
Small-Cap Value	3.0	-3.0
International Stocks	3.0	-3.0
High Yield	2.0	-2.0
Intermediate-term Bonds	1.0	-1.0
Short-term Bonds	1.0	-1.0
Cash Equivalents	0.5	-0.5

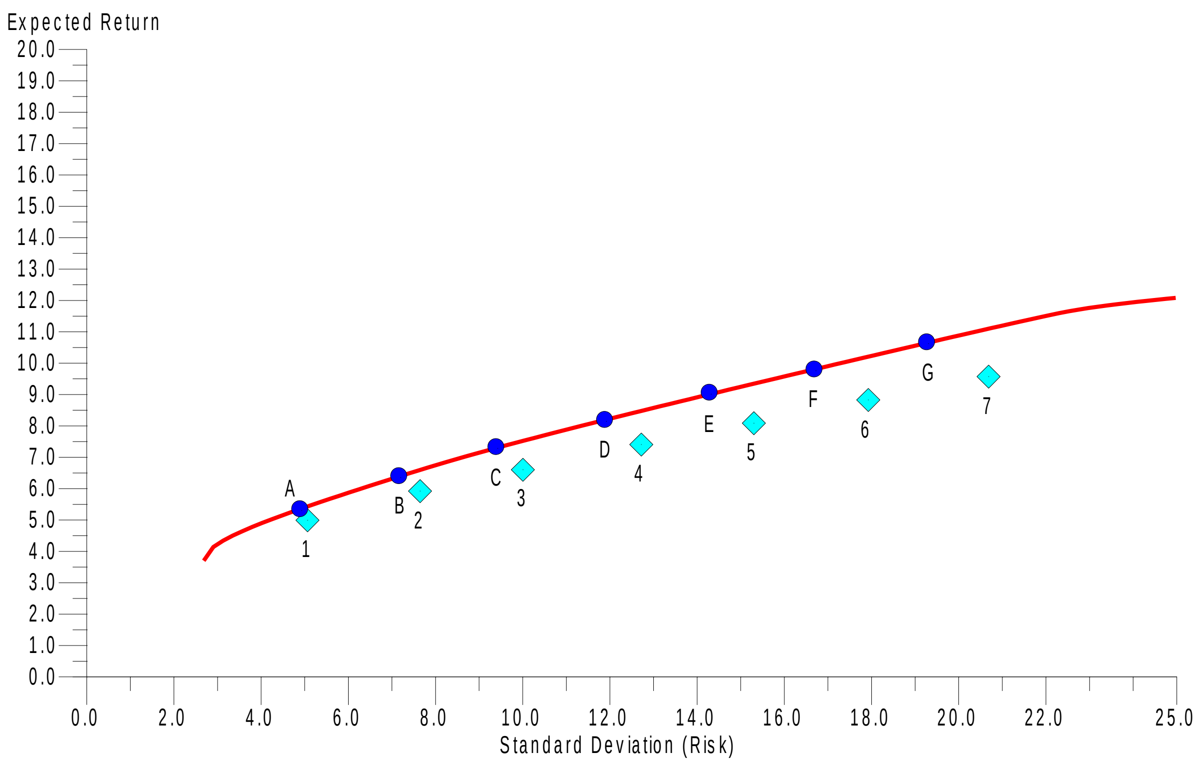
Correlation coefficients remain unchanged



Scenario 3: Expected returns decrease and standard deviations increase for domestic equity asset classes

Asset Class	Change in E(R)	Change in Standard Deviation
Large-Cap Growth Stocks	-2.0	2.0
Large-Cap Value Stocks	-2.0	2.0
Mid-Cap Stocks	-2.5	2.5
Small-Cap Growth	-3.0	3.0
Small-Cap Value	-3.0	3.0
International Stocks	-	-
High Yield	-	-
Intermediate-term Bonds	-	-
Short-term Bonds	-	-
Cash Equivalents	-	-

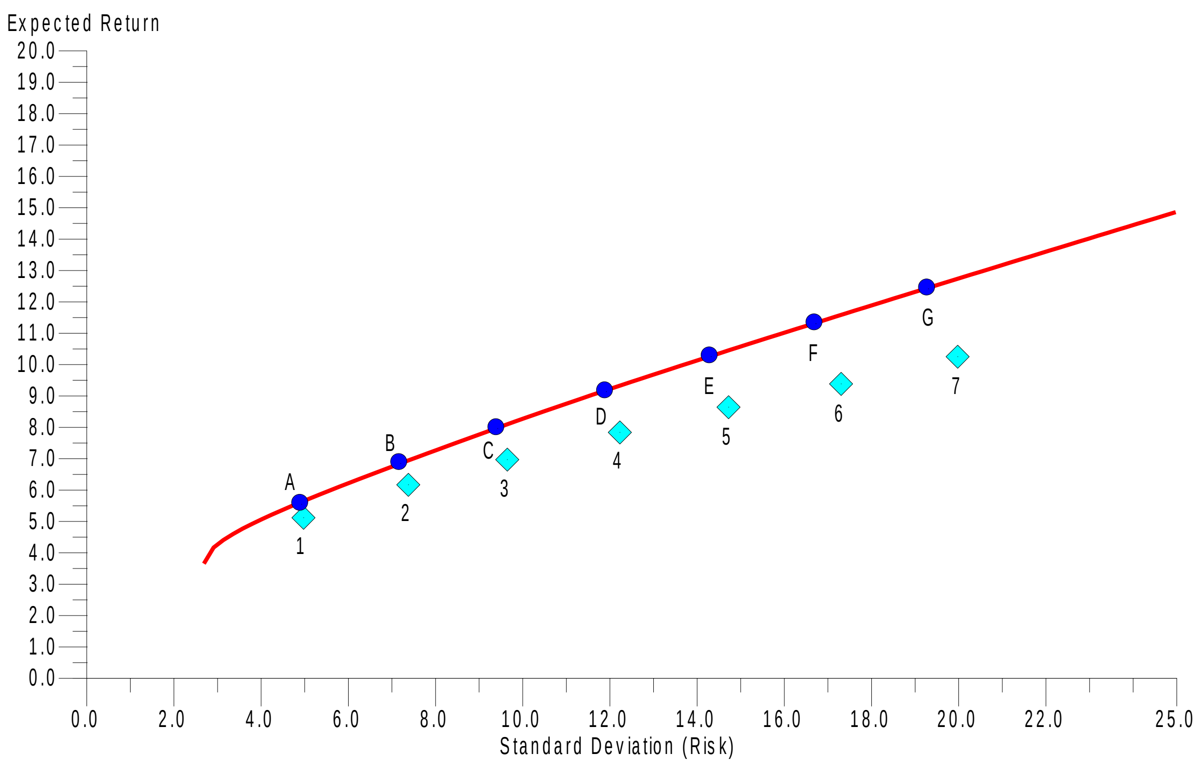
Correlation coefficients remain unchanged



Scenario 4: Expected returns decrease and standard deviations increase for the international equity asset class

Asset Class	Change in E(R)	Change in Standard Deviation
Large-Cap Growth Stocks	-	-
Large-Cap Value Stocks	-	-
Mid-Cap Stocks	-	-
Small-Cap Growth	-	-
Small-Cap Value	-	-
International Stocks	-3.0	3.0
High Yield	-	-
Intermediate-term Bonds	-	-
Short-term Bonds	-	-
Cash Equivalents	-	-

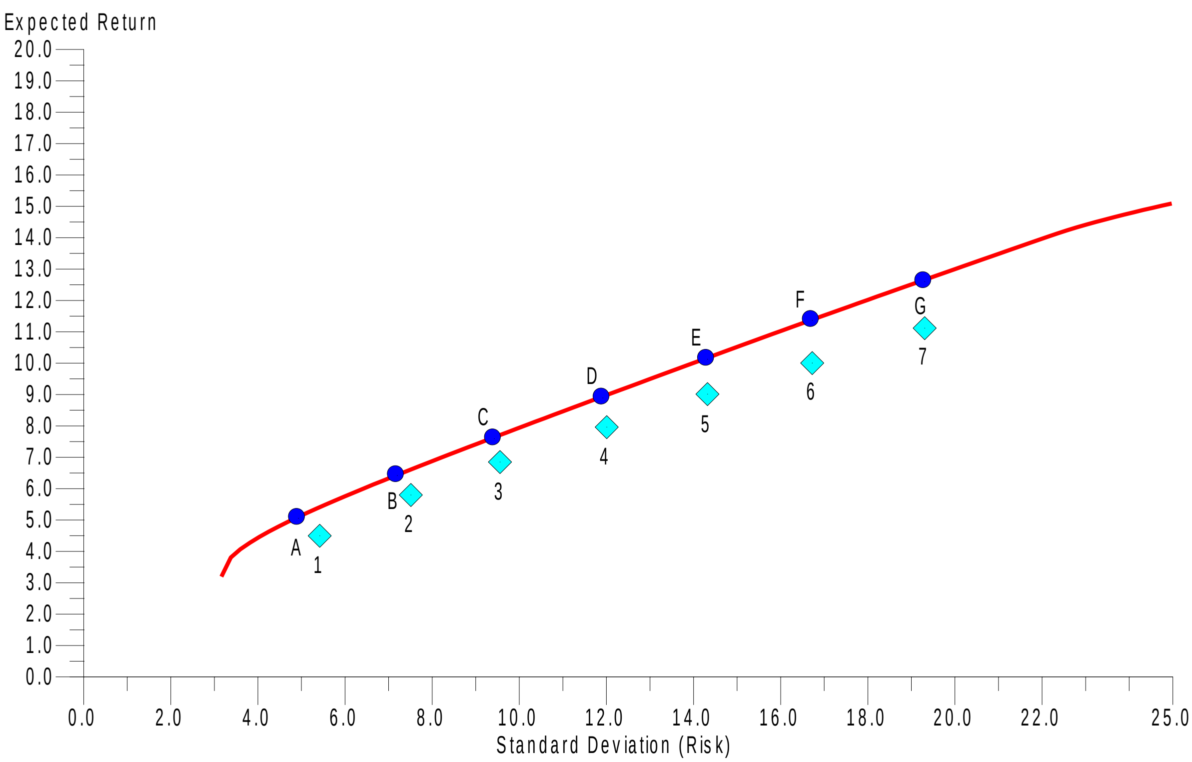
Correlation coefficients remain unchanged



Scenario 5: Expected returns decrease and standard deviations increase for fixed income asset classes

Asset Class	Change in E(R)	Change in Standard Deviation
Large-Cap Growth Stocks	-	-
Large-Cap Value Stocks	-	-
Mid-Cap Stocks	-	-
Small-Cap Growth	-	-
Small-Cap Value	-	-
International Stocks	-	-
High Yield	-2.0	2.0
Intermediate-term Bonds	-1.0	1.0
Short-term Bonds	-1.0	1.0
Cash Equivalents	-0.5	0.5

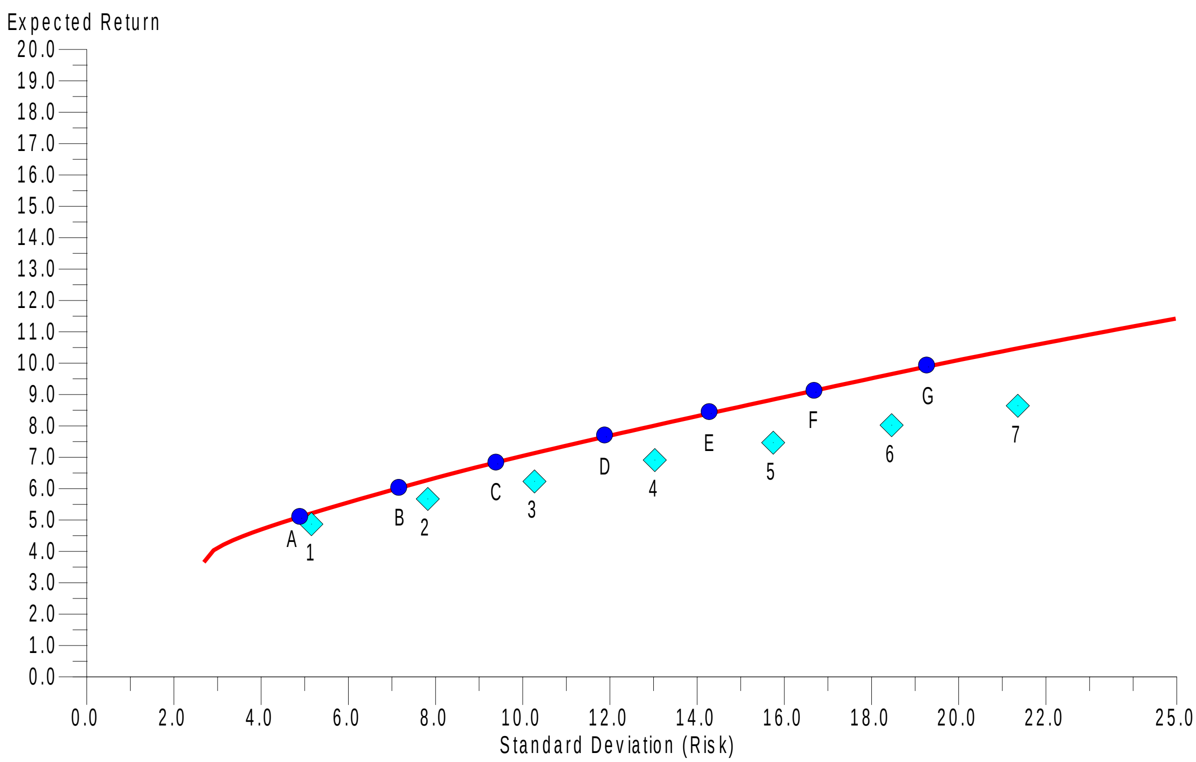
Correlation coefficients remain unchanged



Scenario 6: Expected returns decrease and standard deviations increase for all equity asset classes.

Asset Class	Change in E(R)	Change in Standard Deviation
Large-Cap Growth Stocks	-2.0	2.0
Large-Cap Value Stocks	-2.0	2.0
Mid-Cap Stocks	-2.5	2.5
Small-Cap Growth	-3.0	3.0
Small-Cap Value	-3.0	3.0
International Stocks	-3.0	3.0
High Yield	-	-
Intermediate-term Bonds	-	-
Short-term Bonds	-	-
Cash Equivalents	-	-

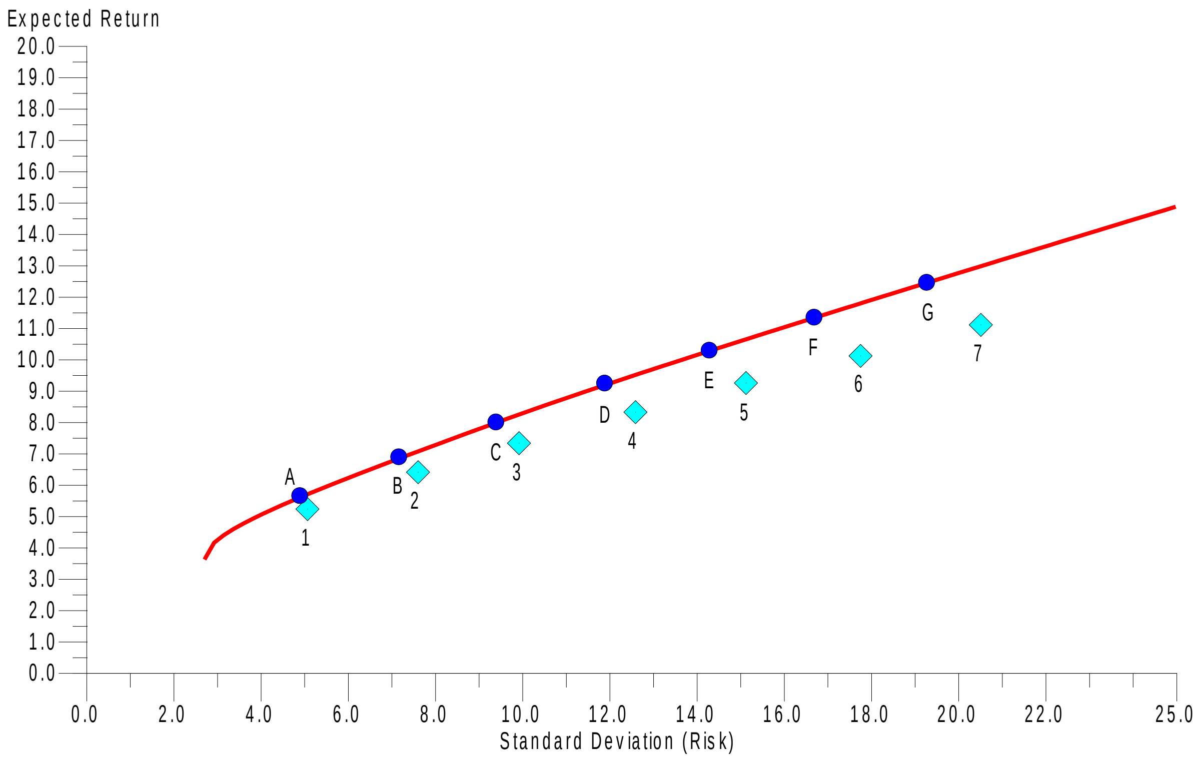
Correlation coefficients remain unchanged



Scenario 7: International and domestic securities exhibit higher correlation coefficients with each other.

Change in correlation coefficients:

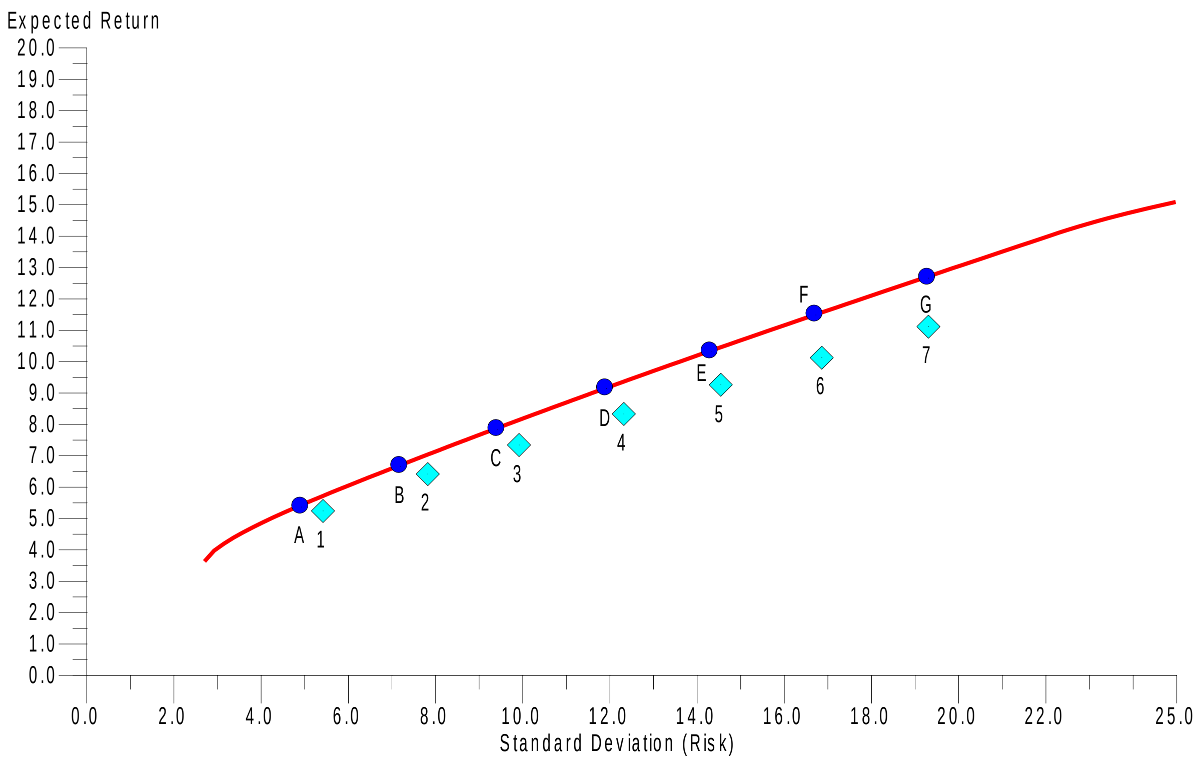
Asset Class	Large-Cap Growth	Large-Cap Value	Mid-Cap	Small-Cap Growth	Small-Cap Value	Intl	High Yield	IT Bonds	ST Bonds	Cash
Large-Cap Growth Stocks	1									
Large-Cap Value Stocks		1								
Mid-Cap Stocks			1							
Small-Cap Growth				1						
Small-Cap Value					1					
International Stocks	0.2	0.2	0.2	0.2	0.2	1				
High Yield						0.1	1			
Intermediate-term Bonds						0.1		1		
Short-term Bonds						0.1			1	
Cash Equivalents						0.1				1



Scenario 8: Equity and fixed income securities exhibit higher correlation coefficients with each other.

Change in correlation coefficients:

Asset Class	Large-Cap Growth	Large-Cap Value	Mid-Cap	Small-Cap Growth	Small-Cap Value	Intl	High Yield	IT Bonds	ST Bonds	Cash
Large-Cap Growth Stocks	1									
Large-Cap Value Stocks		1								
Mid-Cap Stocks			1							
Small-Cap Growth				1						
Small-Cap Value					1					
International Stocks						1				
High Yield	0.2	0.2	0.2	0.2	0.2	0.2	1			
Intermediate-term Bonds	0.2	0.2	0.2	0.2	0.2	0.2		1		
Short-term Bonds	0.2	0.2	0.2	0.2	0.2	0.2			1	
Cash Equivalents	0.2	0.2	0.2	0.2	0.2	0.2				1



Section III – Appendices

Appendix A – Benchmark Explanations

A) CRSP Deciles

The CRSP Deciles are market value weighted benchmarks of common stock performance provided by the Center for Research in Security Prices at the University of Chicago Graduate School of Business.

The CRSP universe includes common stocks listed on the NYSE, AMEX, and the NASDAQ National Market excluding the following: preferred stocks, unit investment trusts, closed-end funds, real estate investment trusts, americus trusts, foreign stocks, and American depository receipts.

All eligible companies listed on the NYSE are ranked by market capitalization on the last trading day of each quarter, then split into ten equally populated groups, or deciles. The capitalization of the largest company in each decile serves as a breakpoint for that decile. When multiple issues of a company are traded, the sum of the issues' capitalization is used for the company capitalization so that each company is counted only once. The portfolios are reformed every quarter using the price and shares outstanding at the end of the previous quarter. During the quarter, companies move between deciles, since their market capitalization changes while the breakpoints of each decile remain fixed.

Decile returns are value weighted and calculated monthly. Security weights are determined using market capitalization based on the shares outstanding and closing price for the last trading day of the previous month. Dividends and split factors are included in the month containing the ex-dividend date. Certain distributions such as spin-offs and rights are reinvested on the ex-dividend date.

Breakpoints are based exclusively on companies with issues traded on the NYSE. For the CRSP series that include securities from the AMEX and NASDAQ over-the-counter market (these are the series used in this project) non-NYSE companies are assigned to appropriate portfolios according to their capitalization in relation to the NYSE decile breakpoints. Thus, the series that include non-NYSE securities are not comprised of true deciles in the sense that an equal number of companies are represented in each of the ten portfolios.

The following tables provide the CRSP Decile breakpoints as of December 31, 2002:

Portfolio Breakpoints (NYSE)

CRSP Portfolio	Number of Companies	Lowest Capitalization	Highest Capitalization
1	150	$9,795,216,000	$245,254,456,000
2	150	$4,075,682,000	$9,660,186,000
3	149	$2,115,584,000	$4,074,534,000
4	150	$1,381,266,000	$2,112,730,000
5	150	$930,449,000	$1,371,553,000
6	150	$630,449,000	$928,931,000
7	150	$403,145,000	$629,561,000
8	149	$244,949,000	$401,324,000
9	150	$97,606,000	$244,697,000
10	150	$7,488,000	$97,222,000

The NYSE portfolio breakpoints are applied to all other markets to incorporate securities listed on the AMEX and NASDAQ exchanges.

Portfolio Breakpoints (NYSE, AMEX & NASDAQ)[11]

CRSP Portfolio	Number of Companies	Lowest Capitalization	Highest Capitalization
1	162	$9,795,216,000	$245,254,456,000
2	190	$4,075,682,000	$9,660,186,000
3	198	$2,115,584,000	$4,074,534,000
4	202	$1,372,134,000	$2,112,730,000
5	225	$930,449,000	$1,371,553,000
6	268	$630,142,000	$928,931,000
7	360	$401,991,000	$629,561,000
8	384	$244,949,000	$401,324,000
9	771	$97,386,000	$244,697,000
10	1833	$269,000	$97,222,000

[11] Center for Research in Security Prices, University of Chicago, December 31, 2002.

B) CFI Bonds

Ibbotson uses Colman, Fisher, and Ibbotson (CFI) estimate bond data in order to calculate a number of fixed income forecasts. CFI estimated government bond methodology developed by Thomas Coleman, Lawrence Fisher, and Roger Ibbotson uses historical data on US Treasury securities to develop yield curves for varying maturity bonds.

The CFI methodology uses a nonlinear least squared (NLLS) regression method to estimate forward rates which best fit observed, historical prices. Coleman, Fisher, and Ibbotson assume that forward rates are constant over certain intervals in order to simplify the model.

The calculation period uses the exact dates of payments and days to payments rather than a simple semiannual time frame. The length of the forward rate period varies over the yield curve because yields tend to more variable at the short-end as opposed to the long-end of the curve. Forward periods also tend to vary because the last trading day of a month does not always fall on the last day of the month. Thus, the first six months may contain between 181 and 187 days.

The following table outlines the various time periods used to estimate forward rates:

Forward Rate Periods					
	Period	Span		Period	Span
Period 1[a]	week 1	7 days	**Period 8**	91 days - 6 mos.	92 days
Period 2[a]	week 2	7 days	**Period 9**[a]	6-12 months	6 months
Period 3[a]	week 3	7 days	**Period 10**	1-2 years	1 year
Period 4[a]	week 4	7 days	**Period 11**	2-4 years	2 years
Period 5[a]	week 5	7 days	**Period 12**	4-8 years	4 years
Period 6[a]	36-50 days	15 days	**Period 13**	8-16 years	8 years
Period 7	51-90 days	40 days	**Period 14**	16-32 years	16 years

a Periods 1-5 depend on the day that a Treasury bill matures. Period 1 may vary from 4 to 10 days. Period 6 may begin from 33 to 39 days from the quotation date. Period 9 ends on the last day of the twelfth month after the quotation date.

The NLLS method used by Coleman, Fisher, and Ibbotson attempts to balance price error and yield error. Price error occurs when the price of the bond is misreported or misrepresented. Although bond prices are typically quoted in terms of bid or ask, the "true" price of the bond is somewhere between these two values. A large bid/ask spread would indicate a wide range of potential "true" prices. In order to minimize this type of error, bonds with small bid/ask spreads are weighted more heavily than bonds with large bid/ask spreads. Bonds that mature within a certain period are weighted by an inverse function of the bid/ask spread.[12]

Yield error is price error divided by the bond's duration. Misquoted prices affect short-term bonds considerably more than long-term bonds. An error of $1 in the price of a 90-day T-bill translates into an

[12] For more information on Coleman, Fisher, Ibbotson yield curves, please see the following:

Thomas S. Coleman, Lawrence Fisher, and Roger G. Ibbotson. (1993). *Historical U.S. Treasury Yield Curves.* Ibbotson and Moody's Investors Service.

Thomas S. Coleman, Lawrence Fisher, and Roger G. Ibbotson. "Estimating the Term Structure of Interest Rates from Data that Include the Prices of Coupon Bonds," *The Journal of Fixed Income,* September 1992.

error of at least 400 basis points when measuring the bond's yield. A similar error in the price of a long-term bond with a ten-year duration results in a yield error of only 10 basis points. Small errors in the prices of short-term bonds have a large effect on the yield.

The equation used by Coleman, Fisher, and Ibbotson attempts to minimize both price and yield error. Their criterion for calculation is given below:

The NLLS Method $\min\left[\dfrac{\sum_{i=1}^{n}\dfrac{B_i^2\left(P_i - P_i^*\right)^2}{D_i}}{\sum_{i=1}^{n}\dfrac{B_i^2}{D_i}}\right]$

Where	P_i	actual price
	P_i^*	predicted price
	D_i	bond duration
	B_i^2	inverse function of the bid/ask spread

Once the forward rate is known, the estimated yields can be used to find spot rates, discount factors, and the implied rates or present values for any type of government fixed income security. The appropriate term structure can be used to compute the expected returns and standard deviations over a specified holding period once the specific characteristics of the bond are determined (i.e. maturity, tax status, and coupon frequency).

This project incorporates a number of CFI estimated bonds. All of these series are coupon at par bonds that are fully taxable with a semiannual coupon frequency.

Appendix B – Input Clarifications: Proxies

When encountering a benchmark with data history shorter than the relevant time period (1926 for stocks and 1970 for bonds) it is necessary to extend the benchmark, either by substituting a similar proxy or by extending the relationship between a proxy and the short-lived benchmark. Ibbotson determines the proxies for short-lived benchmarks by developing portfolios of CRSP deciles (for stocks) or CFI bonds (for bonds). The portfolios of CRSP deciles or CFI bonds will have similar behavior to the short-lived benchmark over their common period.

The proxy benchmarks are determined using returns-based style analysis. The results of this analysis provides a portfolio which best fits the returns of the benchmark. Returns-based style analysis is a quadratic programming approach that is similar to a statistical regression but with two constraints:

- All coefficients must sum to 100 percent
- No coefficients can be negative

Negative coefficients can be interpreted as short positions in asset classes. This type of strategy is not used by indexes and prohibiting these coefficients provides better, more usable results. The object of this approach is to compare returns to a comprehensive set of market benchmarks over some period of time. Ideally, the set of benchmarks should fully reflect the investing universe and be mutually exclusive.

The returns-based style analysis results for the proxy of short-lived stock and bond benchmarks are listed below. The following results represent the overall style over the entire history (since inception of the index) of the asset class benchmark.

Stocks

	CRSP Deciles				R^2
	1–2	3–5	6–8	9–10	
S&P SmallCap 600		36	64		93

Fixed Income

	1 Year CFI	2 Year CFI	3 Year CFI	5 Year CFI	10 Year CFI	20 Year CFI	R^2	Benchmark start date
LB IT Gvt/Corp		60		33	7		96	1973
LB 1-3 Yr Gvt	61		33	6			97	1976

*- The CFI bonds used in the returns-based style analysis process vary, based on the general nature of the asset class. For example, a long-term fixed income asset class benchmark is compared to one set of CFI bonds, while a short-term asset class benchmark is compared to another. The exposure weights obtained from the return-based style analysis are used to create a portfolio of CFI government bonds. This synthetic portfolio is then used as the proxy for the asset class benchmark.

As mentioned in the report, the *current* maturity is used for the high yield and cash equivalents asset classes. In lieu of this analysis, a CFI bond with the same current maturity as the asset class benchmark is used.

Asset Class	Benchmark	Current Maturity	Government Bond Proxy Maturity
High Yield	LB High Yield	8 Years 3 Months	CFI 8 Year 3 Month Bond
Cash Equivalents	CG U.S. 3 Month T-Bill	90 Days	CFI 90-Day Bill

Appendix C – Statistical Methods

This appendix outlines the statistical methods used by Ibbotson to calculate arithmetic mean returns, standard deviations, and correlation coefficients. The standard statistical methods typically used for data over consistent units of time are also included.

Whenever possible, Ibbotson uses annual data in the calculation of inputs. For indexes or return series that have less than a 20-year history, however, monthly or quarterly data is used in order to ensure a sufficient number of data points.

All data used in this project includes data current through year-end 2002. The commentary below describes how annual data has been incorporated into the estimation of optimization inputs.

A) Arithmetic Mean Return
The arithmetic mean return is the simple average of all the returns in a given period. To calculate the annual arithmetic mean, we need the returns.

$R_{(x)}(1), R_{(x)}(2), R_{(x)}(3), R_{(x)}(4)....R_{(x)}(t)$

Where:

$R_{(x)}$ is the return in year t for asset X

The arithmetic mean then, is the sum of all the returns divided by the total number of returns.

$$\text{Mean } \mu_{x^*} = \frac{\sum_{t=1}^{n} R_x(t)}{n}$$

Where:

$R_x(t)$	is the return in year t
n	is the number of years in the period being measured
μ_{x^*}	is the annual arithmetic mean return

B) Geometric Mean Return

The geometric mean of a return series over a period is the compound rate of return over the period. The geometric mean return equation is as follows:

Geometric Mean Return:

$$\text{Mean } \mu_{x^*} = \left[\prod_{t=1}^{n}(1+R_x(t))\right]^{1/n} - 1$$

Where:

$R_x(t)$	is the return in year t
n	is the number of years in the period being measured
μ_{x^*}	is the annual geometric mean return

C) Standard Deviation

The standard deviation of a typical series is the square root of the sum of the squared differences between each return and the mean, divided by the number of data points less 1.

$$\text{Standard Deviation} \quad \sigma_x = \sqrt{\frac{\sum_{t=1}^{n}[R_x(t)-\mu_{x^*}]^2}{n-1}}$$

Where:

$R_x(t)$	is the return in year t
n	is the number of years in the period being measured
μ_{x^*}	is the annual arithmetic mean return

D) Correlation Coefficients

Typically, the correlation coefficient between two series is determined if the covariance between the two series is calculated and the standard deviations of each are known. To find the covariance of two series, the cross product between series X and series Y must first be determined. The cross product is the product of the differences between all X and Y values and their respective means.

Cross Product $SXY = \sum_{t=1}^{n}[R_x(t) - \mu_{x*}] \times [R_y(t) - \mu_{y*}]$

Where:

$R_x(t)$	is the return in year t
μ_{x*}	is the arithmetic mean return of asset X
$R_y(t)$	is the return on asset Y in period t
μ_{y*}	is the arithmetic mean return of asset Y

The cross product is used in the calculation of the covariance. The covariance is the cross product of X and Y divided by the number of annual periods minus 1 (n-1).

Covariance Cov (X,Y) = $\frac{SXY}{n-1}$

The covariance is used in the calculation of the correlation coefficient. The correlation coefficient is the linear association between two variables. The coefficient lies between the values of +1 and -1. A correlation of +1 indicates a perfect positive association and -1 a perfect negative association.

Correlation Coefficient $\rho_{x,y} = \frac{\text{Cov (X, Y)}}{\sigma_{\text{x}} \times \sigma_y}$

Where:

σ_x	is the standard deviation of asset X
σ_y	is the standard deviation of asset Y
Cov (X,Y)	is the covariance between asset X and asset Y

Appendix D – Correlation Coefficient Extension Process

Ibbotson uses a statistical procedure, described in this appendix, to estimate extended correlations among benchmarks and to develop adjusted historical returns and standard deviations for domestic short-history asset classes. The purpose of this procedure is to use additional available information to model the behavior of "short-history" benchmarks, those benchmarks for which we have less baseline data than for some "long-history benchmarks." The additional information in long-history benchmarks judged to be predictive of a short-history benchmark is incorporated into parameter input estimates through this process.

In addition to the problem of short-history benchmarks, the process of developing a consistent estimate for correlations between asset class benchmarks faces an additional challenge particular to the Ibbotson methodology. Ibbotson uses data going back to 1926 in developing expectations for future equity performance, and data going back to 1970 for fixed income analysis. Thus, correlations between fixed and equity benchmarks can only be based on data since 1970. Correlations among equity benchmarks, however, we believe, should be based on the longest time period available.

The idea behind the extension approach is to model each short-history benchmark as an optimal mix of long-history benchmarks. A statistical model is used to determine this mix. These models are used with the correlations between long-history benchmarks to imply the correlations between short- and long-history benchmarks. Further, the models are used imply one component of the correlations between two short-history benchmarks. The second component of the correlations between short-history benchmarks is determined by the residuals from the modeling process.

This appendix describes the extension process and considers its relationship to other suggested methods of adjusting historical data. Subsection 1 describes the steps of the extension process in more detail. Subsection 2 compares the Ibbotson extension process to some other methods that are used to adjust historical estimates. Subsection 3 provides a technical description of the extension procedure.

1. Conceptual Description of the Extension Process

Each short-history benchmark is modeled by at least one long history benchmark. The model is a simple linear model where the returns of the short-history benchmark are represented as a weighted linear combination of the returns of the long-history benchmarks plus a constant. The constant and the weights are determined by regression analysis. A technical description of the method is provided in Subsection 3(c).

The statistical model can be used to adjust a short-history benchmark's expected return. This is done by using the model to forecast returns for periods when the short-history benchmark did not exist. This is done by applying the weights and constant determined by the model to the long-history benchmarks. The actual and forecast benchmarks are then averaged together. A technical description of the method is provided in Subsection 3(d).

Extended correlations are derived from an extended covariance matrix. The extended covariance matrix is constructed using the covariance matrix of the long-history benchmarks and the covariance matrix of the residuals from the model building process. The calculation of the covariance matrix of long-history benchmarks is a standard procedure and is described in Subsection 3(e). The residuals from the modeling process of Subsection 3(c) are important information regarding the nature of the departure of historical returns from the statistical model of those returns. The covariance of these residuals identifies correlations between short-history benchmarks that cannot be captured through the statistical models. Subsection 3(f) introduces notation relating to computing this matrix.

The extended covariance matrix is computed by combining the two covariance matrices with the statistical models. This process is explained in Subsection 3(g). The statistical models represent short-history benchmarks as linear combinations of long-history benchmarks. The covariances between long-history benchmarks then imply the covariances between short- and long-history benchmarks. The statistical models also imply a component of the covariances between the short-history benchmarks themselves. Finally, the extended correlation matrix is easily determined from the extended covariance matrix.

If the statistical models of the short-history benchmarks were perfect, then the covariances between them could be perfectly determined from the covariances between the long-history benchmarks that are used to model them. Similarly, covariances between short- and long-history benchmarks could also be perfectly inferred from the models. Since the models, however, are only approximations, the residuals from the modeling process must be incorporated into the estimation process as well.

2. Relationship to Other Methods

There are other methods of adjusting historical inputs. Shrinkage estimators, pioneered by Stein (1955), are a large class of such methods. The basic idea of shrinkage estimation is to adjust the mean of a subpopulation toward the mean of a larger, encompassing, population. A relevant example is that the expected return of a small-cap benchmark could be estimated by adjusting the observed historical return in the direction of the mean return for the market as a whole. This type of adjustment of optimization inputs is advocated by Jorion (1986), DiBartolomeo (1991), and Michaud (1998). The Ibbotson extension process has some similarities to and some differences with shrinkage estimation. This subsection provides a brief explanation of why we believe the extension process described here is a more appropriate way to generate optimization inputs.

Shrinkage estimation has some relation to the phenomenon of regression to the mean. Sports statistics provide a common example of regression to the mean. The difference between the best and worst performers can ordinarily be expected to decrease over the course of a season. If this is true, then adjusting early-season performance statistics toward the then-observed mean may provide a better prediction of end-of-season performance than simple averages. Shrinkage estimators provide methods for making this type of adjustment.

When applied to the generation of optimization inputs, shrinkage estimators will tend to adjust performance of benchmarks in the direction of overall market performance. This will reduce the extremes in expected returns, standard deviations, and correlations. The degree of adjustment will be related to the uncertainty associated with a benchmark relative to the market. For example, using the Stein (1955) shrinkage estimator, the degree of adjustment of a benchmark toward the expected market return will be proportional to the variance of the benchmark relative to the variance among benchmarks.

The Ibbotson method can be described in this framework as follows. First, instead of adjusting toward overall market behavior, adjustment is made toward a composite benchmark represented by a model or set of models described by equation (1). Then, instead of making the degree of adjustment proportional to a measure such as the variance of the short-history benchmark, the degree of adjustment essentially becomes proportional to the explanatory power (i.e., R^2) of the model. This is a different type of mechanism since the explanatory power of the model is not necessarily related to the variance of the short-history benchmark. Indeed, to the degree that they are related (e.g., higher variance benchmarks might be associated with lower model R^2 values), the Ibbotson adjustment operates on an opposing principle. The Stein estimator will tend to adjust more toward the mean under a high variance/low R^2 scenario, while the Ibbotson method will adjust less toward the composite benchmark implied by the model. Similarly, under a low variance/high R^2 scenario the Ibbotson method will adjust more toward the composite benchmark, while the Stein estimator will adjust less toward the market return.

The most important advantage of the Ibbotson approach is that it directly addresses the basic problem of the incorporation of information contained in long-history benchmarks into expectations for short-history benchmarks. This stands in distinction to the basic problem addressed by shrinkage estimation as described in the beginning of this subsection. In our judgment, the adjustment to represent an appropriate historical record is more fundamental and of larger absolute magnitude than adjustments to account for phenomena such as regression to the mean.

A further advantage of the Ibbotson method is that professional judgment enters into model development in a more understandable way, principally through the construction of the models of short-history benchmarks. Shrinkage estimators have varying and complex estimation methods. Choosing an appropriate estimator is a nontrivial task. Often, these estimators are Bayesian. This type of estimator requires the specification of appropriate prior beliefs. Making such probability assessments can be a difficult undertaking.

3. Detailed Description of the Extension Procedure

3a. Basic Notation and Matrix Algebra

The extension method is most easily described using matrix algebra. A matrix **X** is a group of numbers ordered in a rectangular grid. The element $\mathbf{X_{ij}}$ is the number found at the intersection of the i^{th} row and the j^{th} column. If **X** has *m* rows and *n* columns, it has dimensions $(m \times n)$**.** A row vector is a matrix with only one row, and a column vector has only one column. To add two matrices, they must have the same dimensions and corresponding elements are added together: If $\mathbf{Z = X + Y}$, then $\mathbf{Z_{ij} = X_{ij} + Y_{ij}}$. To multiply two matrices, the column dimension of the first matrix must be equal to the row dimension of the second matrix. If **X** is $(m \times n)$, **Y** is $(n \times r)$, and $\mathbf{Z = XY}$, then **Z** has dimension $(m \times r)$. Each element $\mathbf{Z_{ij}}$ of **Z** is computed according to the formula

$$\boldsymbol{Z}_{ij} = \sum_{k=1}^{n} \boldsymbol{X}_{ik} \boldsymbol{Y}_{kj} \ .$$

The matrix $\boldsymbol{X}'$ is the transpose of $\boldsymbol{X}$. This means the rows and columns are exchanged, so that $X'_{ij} = X_{ji}$. If $\boldsymbol{X}$ is $(m \times n)$, then $\boldsymbol{X}'$ is $(n \times m)$.

3b. Description of Benchmarks

Represent the set of long-history benchmarks as the matrix $\mathbf{X}$. This matrix has dimensions $T \times n_X$, where *T* is the number of time periods and n_X is the number of long-history benchmarks. Each column vector corresponds to a benchmark. There are $n_{\mathbf{Y}}$ short-history benchmarks $\mathbf{Y}^i$, such that for each i, $\mathbf{Y}_t^i$ has data for $t \in \{\tau_l^i, \tau_l^i + 1, \ldots, \tau_u^i\}$, where $1 \le \tau_l^i$, and $\tau_u^i \le T$.

3c. Models of Short-History Benchmarks

We suppose that the true model for each short-term benchmark $\mathbf{Y}^i$ is

$$\mathbf{Y}^{\boldsymbol{i}} = \mathbf{X}^{\boldsymbol{i}} \boldsymbol{\beta}_{\boldsymbol{i}}' + \boldsymbol{\varepsilon}^{\boldsymbol{i}}, \tag{1}$$

where $\mathbf{X}^{\boldsymbol{i}}$ contains those long-history benchmarks used to model $\mathbf{Y}^i$, and the first column of $\mathbf{X}^{\boldsymbol{i}}$ is $\mathbf{1}_{\boldsymbol{T}}$, a $T \times 1$ column vector of ones. Additionally, we assume that realizations of the error vector $\boldsymbol{\varepsilon}$ are independently and identically distributed with an expected value of zero and a constant variance.
This model may then be estimated for each short-term benchmark $\mathbf{Y}^i$ using ordinary least squares (OLS) regression using data on all time periods *t* for which $\mathbf{Y}_t^i$ is observed. Under the assumptions described, the resulting coefficient estimates will provide the best linear unbiased estimate of the relationship between the short-history benchmark being modeled and the long-history benchmarks used as independent explanatory variables.

3d. Estimating Extended Expected Returns for Short-History Benchmarks

The extended expected returns are estimated by first determining the returns predicted by the statistical models described by equation (1) for the time periods that are relevant for the asset class type (i.e., fixed or equity) and not actually observed and then averaging actual and predicted returns. The predicted return for short-history benchmark *i* at time *t*, $\boldsymbol{Y}_t^{\,i}$ is computed using the formula

$$\boldsymbol{Y}_t^{\,i} = \boldsymbol{X}_t^{\,i} \boldsymbol{\beta}_i', \tag{2}$$

where $\boldsymbol{X}_{\boldsymbol{t}}^{\boldsymbol{i}}$ is a $(1 \times (n_{\mathbf{X}} + 1))$ row vector of long-history benchmark returns at time *t*. The extended expected return $\bar{\boldsymbol{Y}}^{\boldsymbol{i}}$ for short-history benchmark *i* is

$$\bar{\boldsymbol{Y}}^{i} = \frac{1}{n_B}\left(\sum Y_t^{\,i} + \sum \hat{Y}_t^{\,i}\right), \tag{3}$$

where $\boldsymbol{n_B}$ is the number of time periods in the extended benchmark history, the first summation is over observed data, and the second summation is over predicted data. Under the assumptions of Subsection 3(c), equation (3) provides the best linear unbiased estimate of $\bar{\boldsymbol{Y}}^{\boldsymbol{i}}$.

3e. Estimating the Covariance Matrix of Long-History Benchmarks

The covariance matrix for the long-history benchmarks contained in the matrix $\mathbf{X}$ is calculated in the usual way. The matrix formula is

$$\boldsymbol{\Omega}_{\boldsymbol{X}} = \frac{1}{\boldsymbol{T} - 1} \boldsymbol{X}'(\boldsymbol{I}_{\boldsymbol{T}} - \frac{1}{\boldsymbol{T}} \boldsymbol{1}_{\boldsymbol{T}} \boldsymbol{1}_{\boldsymbol{T}}') \boldsymbol{X}, \tag{4}$$

where $\boldsymbol{I}_{\boldsymbol{T}}$ is a $T \times T$ identity matrix and, again, $\boldsymbol{1}_{\boldsymbol{T}}$ is a $T \times 1$ vector of ones. An identity matrix has ones along its diagonal and zeros everywhere else.

3f. Estimating the Covariance Matrix of Short-History Benchmark Model Residuals

Define $\boldsymbol{\Omega}_{\boldsymbol{\varepsilon}}$ to be the sample covariance matrix of the regression residuals of the estimation equations described by equation (1). Normally, these calculations will be made over the common period where all residuals are available; that is:

$$t \in \left\{ \max_{i\in\{1,\ldots,n_Y\}} \{\tau_l^i\}, \min_{i\in\{1,\ldots,n_Y\}} \{\tau_u^i\} \right\}. \qquad (5)$$

Common period covariances may be calculated using equation (4) where the **X** matrix is understood to contain the common period residuals and ***T*** is set equal to the number of common period observations.

3g. Constructing the Extended Covariance and Correlation Matrices

Use $\boldsymbol{\Omega}_x$ and $\boldsymbol{\Omega}_{\boldsymbol{\varepsilon}}$ to define $\boldsymbol{\Omega}_0$ as the $n_{\mathbf{X}} + n_{\mathbf{Y}} \times n_{\mathbf{X}} + n_{\mathbf{Y}}$ matrix

$$\boldsymbol{\Omega}_0 = \begin{pmatrix} \boldsymbol{\Omega}_{\mathbf{X}} & \mathbf{0} \\ \mathbf{0}' & \boldsymbol{\Omega}_{\boldsymbol{\varepsilon}} \end{pmatrix},$$

where $\mathbf{0}$ is the $n_{\mathbf{X}} \times n_{\mathbf{Y}}$ zero matrix.

The $\boldsymbol{n_Y} \times \boldsymbol{n_X}$ matrix **B** is defined as follows. For each $\boldsymbol{i}$ from 1 to $n_{\mathbf{Y}}$, let $\hat{\beta}_i$ be the estimated vector of OLS coefficients from the regression of $\mathbf{Y}^i$ onto $\mathbf{X}$ described by equation (1), except for the constant term, where it is understood that the entire set of $\mathbf{X}$ variables may not have been included in $\mathbf{X}^{\boldsymbol{i}}$. The long-history benchmarks included for any short-history benchmark are those that lead to the best model. The $\hat{\boldsymbol{\beta}}_i$ coefficients on the remaining $\mathbf{X}$ variables are set to zero. The matrix $\boldsymbol{B}$ is then formed by stacking the $\hat{\boldsymbol{\beta}}_{\boldsymbol{i}}$ vectors such that $\boldsymbol{B}_{\boldsymbol{ij}}$ is the j^{th} element of $\hat{\boldsymbol{\beta}}_{\boldsymbol{i}}$.

Define $\boldsymbol{\Delta}$ as the $n_{\mathbf{X}} + n_{\mathbf{Y}} \times n_{\mathbf{X}} + n_{\mathbf{Y}}$ matrix:

$$\boldsymbol{\Delta} = \begin{pmatrix} \boldsymbol{I}_x & \mathbf{0} \\ \boldsymbol{B} & \boldsymbol{I}_Y \end{pmatrix},$$

where the identity matrix $\boldsymbol{I}_X$ is $\boldsymbol{n}_{\mathbf{X}} \times \boldsymbol{n}_{\mathbf{X}}$ and $\boldsymbol{I}_Y$ is $\boldsymbol{n}_Y \times \boldsymbol{n}_Y$.

The extended covariance matrix $\hat{\boldsymbol{\Omega}}$, which contains expected covariances of all benchmarks, is then computed according to the following formula:

$$\hat{\boldsymbol{\Omega}} = \boldsymbol{\Delta}\boldsymbol{\Omega}_0\boldsymbol{\Delta}' = \begin{pmatrix} \boldsymbol{\Omega}_x & \boldsymbol{\Omega}_x B' \\ B\boldsymbol{\Omega}_x & B\boldsymbol{\Omega}_x B' + \boldsymbol{\Omega}_{\boldsymbol{\varepsilon}} \end{pmatrix}. \qquad (6)$$

A correlation matrix Θ may be generated from the covariance matrix by setting each element Θ_{ij} equal to the ratio of the ij^{th} covariance to the square root of the product of diagonal elements *ii* and *jj*:

$$\Theta_{ij} = \hat{\Omega}_{ij} \Big/ \sqrt{\hat{\Omega}_{ii}\hat{\Omega}_{jj}}. \qquad (7)$$

3h. Interpretation of the Extended Covariance and Correlation Matrices

Under the assumptions of Subsection 3(c) of this appendix, $\hat{\Omega}$ is the best linear unbiased estimator of the true, but unknown covariance matrix Ω. Equation 6 shows $\hat{\Omega}$ as a partitioned matrix. The top left section of $\hat{\Omega}$, is simply Ω_ε, the covariances between the long-history benchmarks. The bottom-left and top-right sections are the covariances between the short and long-history benchmarks. These covariances are represented by the product matrix $B\Omega_\varepsilon$. The interpretation of this product matrix is that for any short-history benchmark, its covariance with a long-history benchmark is a weighted sum of the covariances of that long-history benchmark with the long-history benchmarks included in the model of the short-history benchmark. The weights are the corresponding regression coefficients from the model.

The covariances between short-history benchmarks, shown in the bottom-right of the partitioned matrix, have two components. The first component, $B\Omega_\varepsilon B'$, contains the covariances implied by the models of the short-history benchmarks. Since $B\Omega_\varepsilon$ represents the covariances between the short and long-history benchmarks, post-multiplication by B' yields the covariances between short-history benchmarks implied by the extension models. The second component is Ω_ε, the covariance between model residuals.

The better the models, the smaller the residual component Ω_ε will be. On the other hand, if the model is completely uninformative, then $B\Omega_\varepsilon B'$ will be close to zero everywhere and Ω_ε will be close to the sample covariances obtained directly from the short-history benchmarks.

When using common period residuals as described by relation (5), $\hat{\Omega}$ will be positive definite. Sometimes, one or more short-history benchmarks may be short relative to other short-history benchmarks to be extended. In this case, it may be desirable to use all available residuals to compute the covariance between any pair of short-history benchmarks. In this case, $\hat{\Omega}$ may not be positive definite.

3i. Estimation with dual baselines

Input extension with different historical baselines is handled in a straightforward way. Estimation of equation (1) is based on the appropriate baseline. All the benchmarks used to model a short-history benchmark must completely cover the relevant historical baseline, that is, they must be a long-history benchmark relative to the baseline. All extended relationships for short-history benchmarks are relative to the chosen baseline. Thus, expected returns estimated by equations (2) and (3) are relative to the appropriate baseline.

Correlations between long-history benchmarks will ordinarily be done on a pairwise basis, a modification of the procedure described by equation (4), where (4) is applied on a pairwise basis to long-history benchmarks using time periods common to both benchmarks. Thus, correlations between long-lived equity benchmarks may go back to 1926. Correlations between long-history fixed-income benchmarks and between those benchmarks and long-history equity benchmarks will be based on data starting in 1970.

References

DiBartolomeo, Dan. 1991. Estimation Error in Asset Allocation. Boston: Northfield Information Services. (http://www.northinfo.com/papers/pdf/19931221_ optimization_robust.pdf)

Jorion, Philippe. 1986. "Bayes-Stein Estimation for Portfolio Analysis*," Journal of Financial and Quantitative Analysis* 21(3): 279-292.

Michaud, Richard. 1998. *Efficient Asset Management*. Boston: Harvard Business School Press.

Stein, Charles. 1955. "Inadmissibility of the Usual Estimation of the Mean of a Multivariate Normal Distribution," *Proceedings of the Third Berkeley Symposium on Probability and Statistics*. Berkeley: University of California Press.

Appendix E – Supply Side Equity Risk Premium Estimate

Stock Market Returns in the Long Run: Participating in the Real Economy

The Debate over Future Stock Market Returns

The impressive performance of the stock market over the last two decades and the resultant increase in investor expectations have spurred numerous articles that call attention to the historical market return and caution investors about their overly optimistic expectations. The articles point to the return of the stock market over the past two years which was well below its historical average, while the bond market, on the contrary, was more than double its historical average. In fact, many studies are predicting returns that are much lower when compared to the historical average. A few even predict that stocks won't outperform bonds in the future.

Approaches to Calculating the Equity Risk Premium

The expected return on stocks over bonds has been estimated by a number of authors who have utilized a variety of different approaches. Such studies can be categorized into four groups based on the approaches they have taken. The first group of studies derives the equity risk premium from historical returns between stocks and bonds. Supply side models, using fundamental information such as earnings, dividends, or overall productivity, are used by the second group to measure the expected equity risk premium. A third group adopts demand side models that derive the expected returns of equities through the payoff demanded by equity investors for bearing the additional risk. The opinions of financial professionals through broad surveys are relied upon by the fourth and final group.

Ibbotson's supply equity risk premium estimate is based upon the work by Roger G. Ibbotson and Peng Chen.[13] Their work combined the first and second approaches above to arrive at their forecast of the equity risk premium. By proposing a new supply side methodology, the Ibbotson-Chen study challenges current arguments that future returns on stocks over bonds will be negative or close to zero. The results affirm the relationship between the stock market and the overall economy. They also provide implications for investors creating an asset allocation policy between stocks and bonds.

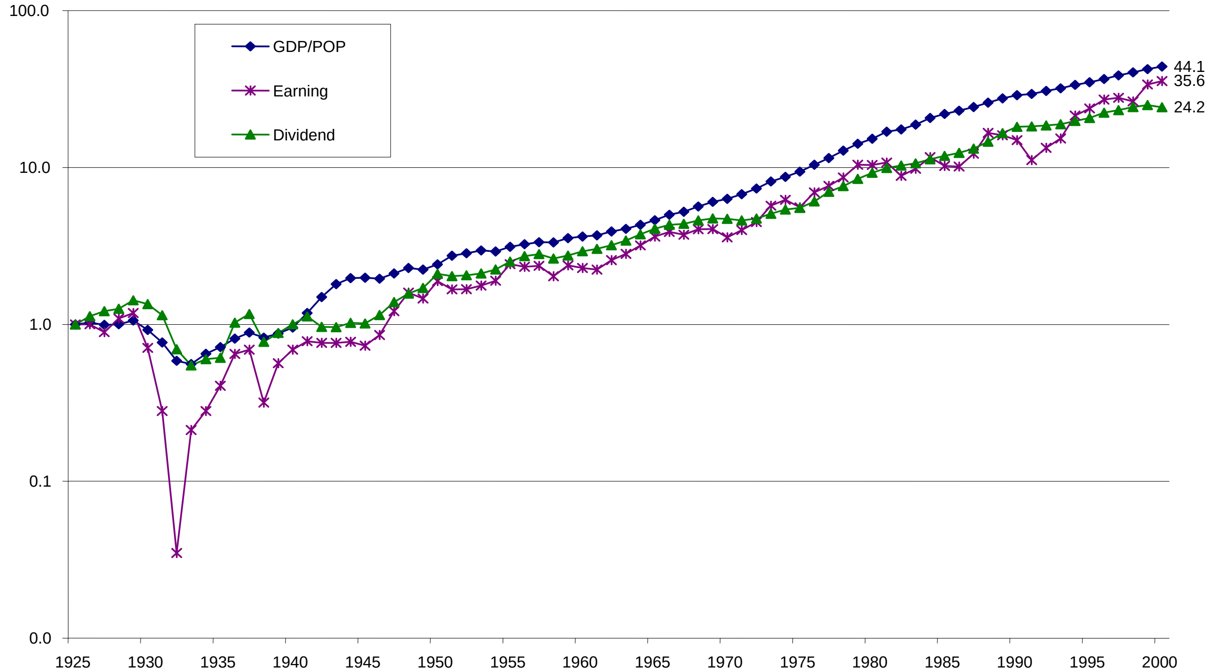
Supply Model

Long-term expected equity returns can be forecast by using supply side models. The supply of stock market returns is generated by the productivity of the corporations in the real economy. Investors should not expect a much higher or lower return than that produced by the companies in the real economy. Thus, over the long run, equity return should be close to the long-run supply estimate.

Earnings, dividends, and capital gains are supplied by corporate productivity. Figure 1 illustrates that earnings and dividends have historically grown in tandem with the overall economy (GDP per capita). However, GDP per capita did not outpace the stock market. This is primarily because the P/E ratio increased 2.54 times during the same period. So, assuming that the economy will continue to grow, all three should continue to grow as well.

[13] Ibbotson, Roger G. and Peng Chen, "Stock Market Returns in the Long Run: Participating in the Real Economy," *Financial Analysts Journal*, January-February 2003.

Figure 1: Growth of $1 at the beginning of 1926
1926-2000

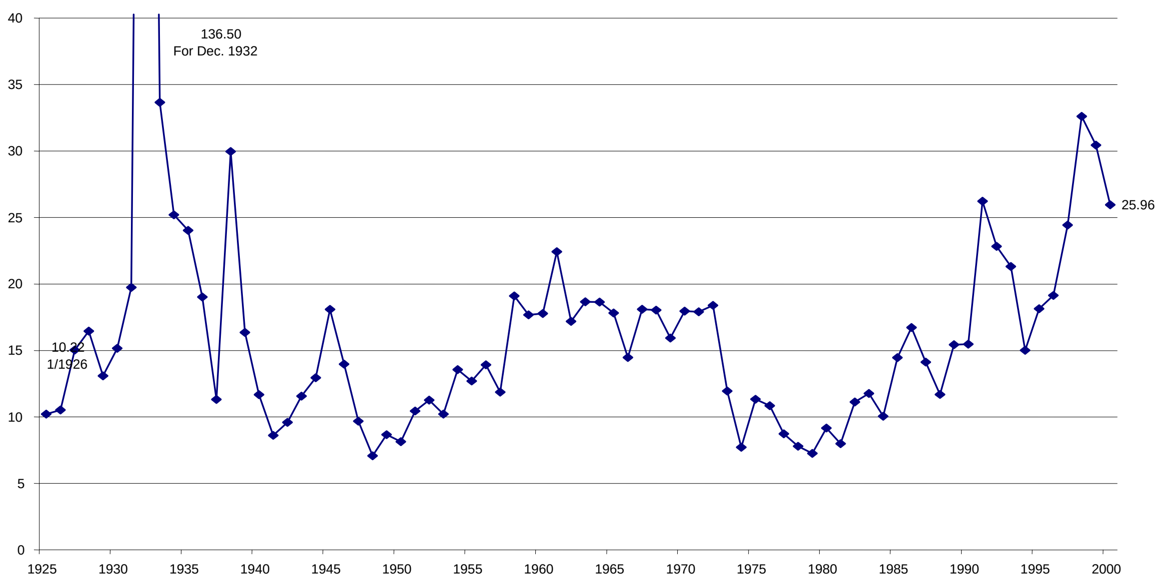


Two main components make up the supply of equity returns: current returns in the form of dividends and long-term productivity growth in the form of capital gains. Two supply side models, the earnings model and the dividend model, are discussed below. The components of the two models are analyzed and those that are tied to the supply of equity returns are identified. Lastly, the long-term sustainable return based on historical information of the supply components is estimated.

Forward-Looking Earnings Model

The earnings model breaks the historical equity return into four pieces, with only three historically being supplied by companies: inflation, income return, and growth in real earnings per share. The growth in the P/E ratio, the fourth piece, is a reflection of investors' changing prediction of future earnings growth. The past supply of corporate growth is forecast to continue; however, a change in investors' predictions is not. P/E rose dramatically over the past 20 years because investors believed that corporate earnings would grow faster in the future. This growth in P/E drove a small portion of the rise in equity returns over the last 20 years. Figure 2 illustrates the price to earnings ratio from 1926 to 2000. The P/E ratio was 10.22 at the beginning of 1926 and ended the year 2000 at 25.96—an average increase of 1.25 percent per year. The highest P/E was 136.50 recorded in 1932, while the lowest was 7.26 recorded in 1979.

Figure 2: P/E Ratio 1926-2000



The historical P/E growth factor is subtracted from the forecast, because we do not believe that P/E will continue to increase in the future. The market serves as the cue. The current P/E ratio is the market's best guess for the future of corporate earnings and there is no reason to believe, at this time, that the market will change its mind. Thus, the supply of equity return only includes inflation, the growth in real earnings per share, and income return.

The arithmetic equity risk premium, based on the supply side earnings model, is calculated to be 5.01 percent—1.21 percentage points lower than the straight historical estimate. Instead of using the one-year earnings in calculating the P/E ratio, as in the Ibbotson and Chen paper, we use the three-year average earnings in this calculation. This is because reported earnings are affected not only by the long-term productivity, but also "one-time" items that do not necessarily have the same consistent impact year after year. Using the three-year average is more reflective of the long-term trend, than the year by year numbers. For example, the 2001 earning used in this calculation is the average reported earnings from 2000, 2001, and 2002. For year 2002, the earning is the average of reported earnings in 2001, 2002, and the estimated earnings in 2003.

Implications for the Investor

For the long-term investor, asset allocation is the primary determinant of returns. Of all the decisions investors make, therefore, the asset allocation decision is the most important.

Asset allocation decisions are determined largely by the equity risk premium—a negative figure would suggest an allocation in favor of fixed income, while a positive number would dictate an allocation in favor of equities (an investor's risk tolerance, investment goals, time horizon, etc., need to be considered as well). But that asset allocation decision is only as good as the accuracy of the investor's forecast of the equity risk premium.

Ibbotson and Chen believe that stocks will continue to provide significant returns over the long run. The equity risk premium, based on the supply side earnings model using 3-year average earnings, is calculated to be 1.21 percentage points lower than the straight historical estimate. The forecast for the market is in line with both the historical supply measures of public corporations (i.e. earnings) and overall economic productivity (GDP per capita).

Appendix F – Forecasting the Inflation Rate

Since 1976, Roger Ibbotson and Rex Sinquefield, and later Ibbotson Associates have provided estimates of the market's long-term forecasts of asset class returns and inflation. The market's forecast of inflation is not directly observable. However, it can be inferred from current yields on Treasury bonds and the statistical time series properties of historical data using techniques first developed by Ibbotson and Sinquefield in "Stocks, Bonds, Bills, and Inflation: Simulations of the Future (1976-2000)" *(Journal of Business*, July 1976). The methodology described below is Ibbotson Associates' most recent refinement of the Ibbotson-Sinquefield methodology as it applies to expected inflation.

A) Theory

The key insight in the analysis presented here is that investors' long-term inflation forecasts are embedded into long-term risk-free yields. Investors expect to be compensated for the lost purchasing power of the dollar over time.

Compensation for expected lost purchasing power is not the only component of Treasury bond yields. Bondholders also expect to be rewarded for foregoing real consumption for a period of time. This reward can be expressed as the expected real risk-free rate.

Bond yields do not remain constant through time. Since holders of long-term bonds typically do not hold their bonds until maturity, the variability of yields is a source of risk for bondholders. Consider an investor with a one-month investment horizon. The investor can purchase either a one-month Treasury bill (and lock-in a return with perfect certainty) or purchase a Treasury bond and face the risk that at the end of the month, the bond's yield will rise (causing a fall in its value). In order for the bond to be an attractive alternative to the bill, the expected return on the bond must be high enough to compensate the investor for the market risk of holding a longer-term instrument. The spread between the expected returns on bonds and bills leads to the embedding of horizon premia into bond yields.

Thus, observed yields on Treasury bonds are composed of three components: expected inflation, expected real risk-free rates, and horizon premia. None of the three components are observable. In order to estimate expected inflation, we estimate expected real risk-free rates and horizon premiums from statistical relationships evident in historical data and remove them from observed market yields. Expected inflation rates are the residuals from this process.

B) Forward Rates

Consider an investor with a 20-year time horizon. The investor can purchase a 20-year zero-coupon bond and lock-in a 20-year return. Alternatively, the investor can purchase a one-year bill and plan on rolling-over the proceeds into another one-year bill, repeating the process for twenty years. If the investor were not concerned about risk, he would be indifferent between these two strategies if they both had the same expected return. Under these conditions, the yield on the 20-year bond must be comprised of the investor's forecast of one-year yields for the next twenty years. This concept can be formalized with forward rates.

The yield on a zero-coupon bond with T years to maturity can be decomposed into T one-year forward rates as follows:

$$Y(T) = \sqrt[T]{[1+F(1)][1+F(2)]\cdots[1+F(T)]} - 1$$

Where: Y(T) = the yield on the bond; and
F(t) = the one-year forward rate which predicts what the yield on a one-year bond will be at the beginning of year t.

If investors were not concerned about risk, each forward rate would consist of a forecast of inflation and the real risk-free rate for its year. Since investors are concerned about risk, each forward rate also includes a horizon premium. To obtain inflation forecasts for each year in the future, we subtract an estimate of the expected real risk-free and horizon premium from each of the forward rates.

C) Continuously Compounded Rates

To simplify calculations, Ibbotson uses continuously compounded rates in all computations. This requires conversion of all variables from discrete rates to continuously compounded rates. Once the continuously compounded rate of inflation is forecast, Ibbotson converts it back to the more familiar discrete form.

The formula for converting a rate of return from discrete rate to continuous rate is:

$$r = \ln(1+R)$$

Where:
r = the continuously compounded rate; and,
R= the discrete rate.

Bond yields are typically expressed in semiannual form. The formula for converting a semi-annual yield to a continuous yield is:

$$y = 2\ln\left(1+\frac{Y}{2}\right)$$

Where:
y = the continuous yield; and
Y = the semiannual yield.

D) Forming the Forecast

The steps to form the forecast of the 20-year inflation rate are as follows:

- Obtain semiannual yields on zero-coupon bonds for maturity between one and twenty years and convert them to continuous yields.
- Derive continuous forward rates from the continuous zero-coupon yields.
- Estimate expected real risk-free rates for the next twenty years.
- Estimate horizon premia for each of the twenty forward rates.
- Subtract the expected real risk-free rates and horizon premia from the forward rates to obtain expected inflation rates expressed in continuous form.
- Combine the expected continuous inflation rates to obtain a forecast for the entire 20-year period.
- Convert the 20-year expected inflation rate from continuous form into discrete form.

Each of these steps is discussed in more detail below.

1) Yields on Zero-Coupon Treasury Bonds

Yields on “Constant Maturity Treasury” rates or CMTs were obtained from the US Treasury as of December 31, 2002. The reported yields for the 1, 2, 3, 5, 7, 10, and 20 year maturities were used. For the remaining maturity periods, yields were filled in by interpolation to obtain a smooth yield curve. The semiannual yields were converted to continuous yields using the formula given above. CMTs are interpolated by the Treasury from the daily yield curve. This curve, which relates the yield on a security to its time to maturity is based on the closing market bid yields on actively traded Treasury securities in the over-the-counter market. These market yields are calculated from composites of quotations obtained by the Federal Reserve Bank of New York. The yield values are read from the yield curve at fixed maturities, currently 3 and 6 months and 1, 2, 3, 5, 7, 10, 20, and 30 years. This method provides a yield for a 10 year maturity, for example, even if no outstanding security has exactly 10 years remaining to maturity. The Treasury yield curve is estimated daily using a cubic spline model. Inputs to the model are primarily bid-side yields for on-the-run Treasury securities.

2) Deriving Forward Rates
The continuous one-year forward rate for year 1 is equal to the continuous zero-coupon yield at one year to maturity. Mathematically,

$$f(1) = y(1)$$

Where:
$f(1)$ = the continuous one-year forward rate for year 1; and,
$y(1)$ = the continuous yield of a zero-coupon bond with a one year to maturity.

The remaining continuous one-year yields are derived by repeated application of the following formula:

$$f(T) = T \cdot y(T) - (T-1) \cdot y(T-1)$$

Where:
T = the year
$f(T)$ = the continuous one-year forward rate for year T;
$y(T)$ = the continuous yield of a zero-coupon bond with T years to maturity;

3) Estimating the Expected Real Risk-Free Rate
The expected real-risk-free rate of return is the constant dollar return an investor expects to receive without taking market risk. The relationship between the short-term real return on bills and inflation (both rates expressed in continuous time) can be described as follows:

$$b(t) = B(t) - \pi(t)$$

Where:
$b(t)$ = the continuous real return on 30-day Treasury bills in year *t*;
$B(t)$ = the continuous annual return on 30-day Treasury bills in year *t*;
$\pi(t)$ = the continuous rate of inflation in year *t.*

The real-risk-free rate and inflation follow trends. They both exhibit high serial correlation, indicating a statistical relationship between this period's and last period's rate. Accordingly, Ibbotson estimates the expected real-risk-free rate from an auto-regression using data from 1970-present:

$$b(t) = \alpha + \beta[b(t-1)] + \varepsilon(t)$$

Where:
$b(t)$ = the continuous real annual return on 30-day Treasury bills in year t;
α = the intercept of the regression, estimated to be 0.40164 percent;
β = the coefficient of the lagged dependent variable, estimated to be 0.702609;
$\varepsilon(t)$ = the error term of the regression.

The forecast of next year’s real return is:

$$E[b(1)] = \alpha + \beta[b(0)]$$

Where:

$b(0)$ = the continuous real annual return on 30-day Treasury bills as of December 31, 2002.

4) Estimating Expected Horizon Premia
Over the period 1926 to 2002, the differences between the historical averages of continuous annual income returns on Treasury bonds and a 30-day Treasury bill are used to estimate horizon premia. Treasury bonds with one, five, and twenty years to maturity are used to estimate the horizon premia for years 1, 5, and 20 respectively. The remaining horizon premia are obtained by performing an interpolation analysis on these three maturity periods.

5) Forming Year-by-Year Inflation Forecasts
For each future year, the expected continuous rate of inflation is estimated as follows:

$$E[\pi(T)] = f(T) - E[b(T)] - E[hp(T)]$$

Where:

$E[\pi(T)]$ = the expected continuous inflation rate in year T;

$E[hp(T)]$ = the expected horizon premium in year T;

and all other variables are as defined earlier.

6) Combining the Year-by-Year Inflation Forecasts
The forecast of the continuous compound rate of inflation over years 1 through T is simply the average of the one-year forecasts:

$$E[\bar{\pi}(T)] = \frac{\sum_{j=1}^{T} E[\pi(j)]}{T}$$

Where:

$E[\bar{\pi}(T)]$ = the compound rate of inflation over year 1 through T.

7) Converting the Continuous Inflation Forecast to Discrete Form
Finally, we need to convert the continuous time forecasts to discreet time in order to use them in the usual fashion:

$$e^{E[\bar{\pi}(T)]} - 1$$

Using the data and procedures described above, Ibbotson obtains the following results:

Expected Inflation

Years from Today	Year-End Date	Inflation Estimate (percent)
1	2003	1.3%
2	2004	1.2%
3	2005	1.3%
4	2006	1.4%
5	2007	1.6%
6	2008	1.8%
7	2009	1.9%
8	2010	2.0%
9	2011	2.0%
10	2012	2.1%
11	2013	2.2%
12	2014	2.2%
13	2015	2.2%
14	2016	2.3%
15	2017	2.3%
16	2018	2.4%
17	2019	2.5%
18	2020	2.5%
19	2021	2.6%
20	2022	2.7%

The values above are the medians of the distributions of future inflation. Strictly speaking, forecasts should be mathematical expectations, not medians. Mathematical expectations are always greater than medians. However, in practice, the median and the expected value for inflation are very close. So while the values in the above table understate expected inflation by a small amount, they are reasonable estimates of expected future inflation. Therefore our best forecast for the compound annual rate inflation over the next twenty years is 2.7 percent.